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By J. M. MONCRIEFF, M. Am. Soc. C. E.

PRESENTED MAY 2D, 1900.

WITH DISCUSSION.

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In developing a theory of column resistance and its resulting formulas some of the more important points requiring to be kept in view are the following:

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In developing a theory of column resistance and its resulting formulas some of the more important points requiring to be kept in view are the following:

1. The theory should be based on correct principles and the formulas should be of correct form, without introducing refinements which have but little practical value or influence on the results.
2. The theory and formulas should have a wide range of application, to cover the conditions met in engineering practice,

and should not be limited to the case of columns under presumably central loads.

3. The reasoning and the formulas should be sufficiently simple to be understood by other than expert mathematicians, and should be suitable for use in every-day engineering practice.
4. The theory must show agreement with, and be a reasonable explanation of, the results of practical experiment.
5. The formulas should in practice be applicable generally to different materials, by simply introducing the values of the ordinary physical constants of strength and stiffness.
6. Empirical factors should be reduced to a minimum.

The simple theory and formulas which are set forth in this paper show a fair compliance with the foregoing requirements, and it is hoped that they may assist the practical engineer to a better understanding of the important subject of column strength.

The underlying principles upon which the reasoning is based are, (1) that a perfectly centered column of perfect material and straightness is an ideal conception seldom or never realized in practice, and (2) that the various disturbing influences preventing the realization are practically all capable, as regards their ultimate effect, of being represented by an equivalent eccentricity of loading.

Any theory based on these principles ought to be identical in its results with the theory of the ideal perfectly centered column of perfect material and straightness, when the factor representing eccentricity is reduced to zero.

The assumption of the principle of equivalent eccentricity receives practical justification in the records of the experiments of James Christie, M. Am. Soc. C. E., the late Charles A. Marshall, M. Am. Soc. C. E., M. Considère, and Professor Bauschinger, each of whom found that the physical axis of resistance in a column did not necessarily coincide with the geometrical axis, and in fact very frequently did not. Each of these experimenters made tests in which he practically felt for the physical axis in order to obtain a higher strength for the column under trial, and found that it was quite possible for a column to show higher strength when apparently loaded eccentrically, as compared with the strength when apparently loaded exactly over

the geometrical axis. Mr. Christie's and Mr. Marshall's principal tests were both, however, centered over the geometrical axis, and their attempts to feel for the physical axis were supplementary.

After the development of the theory and its resulting formulas, the first difficulty in its application to the case of columns under apparently central loading is the value to be assigned to the equivalent eccentricity, and a careful study of the records of nearly all the more important tests of columns was therefore undertaken, with the view of arriving at some idea of what that value should be.

It became at once apparent, from a comparison of the tests by different experimenters, that isolated tests or a set of tests covering only a small range in proportions, as measured by the ratio of length to radius of gyration, or having only a scanty number of tests at each ratio $\frac{l}{r}$, could in themselves afford no reliable basis for use in practical work, and the writer has no hesitation in saying that any general conclusions or formulas derived from such conditions are absolutely misleading.

It also became most clearly evident that any conclusions deduced from experiments must make full allowance for the possible or probable history of the material of the column during its manufacture and during its preparation for the testing machine.

Many of the causes of the divergence of practical experiments from the theoretic ideals have been incidentally alluded to in able papers and discussions on the subject of column resistance, but, very frequently, too much emphasis has been laid upon the variable nature of the material—excepting perhaps in the case of timber—and, on the other hand, too little has been credited to the probable history of the material, and to the influence of apparently insignificant initial curvature in the specimens, or small errors in setting in the testing machine.

The divergence alluded to may, with every probability of truth, be partly credited, in the case of wrought iron and steel, to the effects of the inevitable cold-straightening to which every bar, plate, or shape, turned out of the rolling mill, must be subjected before being fit for use in ordinary construction, and to a still greater degree before being put into a testing machine as a properly prepared specimen.

It cannot be too clearly realized that the material used in every-day construction is in anything but an ideal condition as regards freedom

from internal stresses, and as regards uniformity of elastic resistance in its detailed sections.

This is quite a different thing from assuming that material of similar history and of the same class varies very widely in its compressive strength, or in the value of the modulus of elasticity.

Striking instances of the influence of history have been given by *Sir* Benjamin Baker, Hon. M. Am. Soc. C. E., in the case of experiments which he carried out on solid, mild-steel columns, 30 diameters in length, showing that the resistance varied according to previous treatment, as follows:*

	Tons per square inch.
"Annealed	14.5
Previously stretched, 10 per cent	12.6
" compressed, 8 "	22.1
" " 9 "	28.9
Straightened cold	11.8"

There are also a number of references to the influence of history in Mr. James Christie's† papers on the strength of iron and steel.

The effect of cold-straightening is, of course, to locally strain the material beyond the limit of elasticity, and, without this overstrain, the bar or plate could not be straightened. The result is that at certain points the modulus of elasticity, or, in other words, the stiffness of the fibers overstrained in tension, is lowered very greatly as regards resistance to compressive stress, and the fibers overstrained in compression are affected similarly as regards their resistance to tensile stress. In addition, permanent internal stresses, both tensile and compressive, are set up in the material, and these are neither imaginary nor insignificant.*

A direct consequence of this interference with natural conditions is that the "physical" axis, or the axis passing through the center of resistance of every section of the column, will not be coincident with the geometrical axis, and in forming a mental conception of the physical axis under these artificial conditions, we are driven to the conclusion that in practical work it can rarely, if ever, be a straight line.

If these deductions be extended to the case of short test-specimens under compression, how is it to be expected that accurate determina-

* *Minutes of Proceedings*, Institution of Civil Engineers, Vol. xcii, p. 44.

† *Transactions*, Am. Soc. C. E., Vol. xiii.

tions of the natural elastic compressive strength can be derived from specimens cut from a portion of the material which may previously have been subjected to cold-straightening. Other portions of the same piece may not have been in the straightening press, and it would then be reasonable to expect them to show a higher value of elastic strength.

In the case of built columns, the effect of the process of machine-riveting is another outside influence on the condition of the material which requires recognition. Every practical constructor knows that in riveting up a member by hydraulic machine-riveters, the various parts have a tendency to stretch out and creep past each other, sometimes in very different degrees, resulting in the members twisting or bending out of a straight line, and no clearer evidence can be adduced to prove the existence of somewhat heavy internal stresses in the finished work.

In symmetrical sections the effect of the riveting down one side will be apparently neutralized by the subsequent riveting on the other; but in an assemblage of plates and angle bars, or other sections, as already remarked, the plates and bars often stretch or creep in different degrees, so that, although the resulting member may be quite straight and free from twist, this will be no proof of the non-existence of severe artificial internal stresses.

Among the instances of this which have occurred in the writer's experience, one, in connection with a bridge over the River Tyne, England, may be mentioned.

In order to guard against this creeping tendency, the writer specified that the large columns of the river piers should have their butt-joint ends machined over the full section, after having been riveted up in lengths in the contractors' yard, to ensure that the butts should bear on each other for the full sectional area of the members.

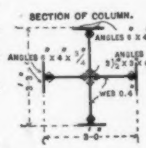


FIG. 1.

These columns are of cruciform section, as shown in Fig. 1, and the total length of each consists of three lengths of about 27 ft. By some oversight, however, in the case of some of the columns, the specified requirement was not carried out in the contractors' yard, and instead, the ends of each individual bar and plate were machined to a very good fit before rivet-

ing was commenced. Tacking rivets were then put in to hold the various parts in their correct relative positions, and the riveting was proceeded with throughout the length of 27 ft., and on its completion it was found that the angle bars and plates had crept past each other in varying degrees, so that the previous careful fitting was of no avail, and the joint had to be re-dressed by hand. The web-plate ends were also found to be hollow to the extent of $\frac{1}{16}$ in. in the half-width of column between the angle bars, and this undoubtedly points to the fact that internal stresses must exist in the column as made.

The columns appeared to be quite straight and true as a whole, and the material of the plates and angles was all made and tested under the same specified requirements of tensile strength, 28 to 32 tons (2 240 lbs.) per square inch, with ultimate elongation of at least 20% in 8 ins. The material is open-hearth acid-process steel, throughout.

In the case of cast-iron columns, of course, cold-straightening or machine-riveters do not enter as disturbing influences, but it is hardly necessary to point out the probably similar influences due to the hidden defects and internal stresses known to exist more or less in all castings.

The other influences to which the writer has referred, *i. e.*, the presence of initial curvature or small errors in setting specimens in the testing machine, have by no means an insignificant value in the results obtained in experiments on "centrally" loaded columns.

Mr. James Christie, in his paper* entitled "Experiments on the Strength of Wrought-Iron Struts," remarked that his specimens were practically straight, but careful measurements revealed the existence of small amounts of initial curvature. These measurements are recorded in his tables of results, and the calculations of deflection hereafter given will show that they had an appreciable influence on the strength of the specimens. Mr. Charles A. Marshall's tests also give valuable evidence in the same direction.

The importance of apparently very small errors in setting the columns in the testing machine also received most definite practical demonstration in the tests by Mr. Christie and Mr. Marshall, and also in those by Professor Bauschinger, since moving the specimens very small distances from their original positions had great influence on the results.

* *Transactions*, Am. Soc. C. E., Vol. xiii.

In view of these prejudicial influences, it can be no matter for surprise that such wide differences exist between results obtained under apparently identical conditions, by the same experimenter, on the same material and with columns of precisely the same proportions. It is surprising, however, that attempts are made constantly to express the average results of such wide differences by a single-line formula.

What would be thought of an engineer who used wrought iron and mild steel indiscriminately, in a bridge or other structure, after striking an average between the tensile strengths of the two materials as a basis upon which to determine the sections of the tension members? This would be no more erroneous and misleading than the present practice of using the average strength of columns of any particular class of material and of any one value of such proportions as are usual in practice. It is surely a much more rational and scientific procedure to ascertain within what limits we may reasonably expect the strength of columns to lie, and then to base our estimate of probable strength on the lower limit so derived.

No reference is here intended to be made to what would be deemed defective columns, in any respect, but only to columns which are in every practical sense believed to be above suspicion. The enunciation of the principle that the strength of columns of any given material cannot be represented by any single-line formula, but must be expressed by an area within which the results of experiments may be expected to lie, was, the writer believes, first made by Professor T. Claxton Fidler, M. Inst. C. E.,* ascribing the variations in column strength to variations in the modulus of elasticity.

In plotting the results of the tests of various experiments upon the diagrams accompanying this paper, the writer has endeavored to show every test wherever possible, provided no defects or special circumstances were involved.

In the case of Professor Tetmajer's tests, the records of a considerable number of the results, as given in his tables, were the averages of two specimens, so that those diagrams on which Tetmajer's tests are plotted do not show the full range of the tests, and the differences between maximum and minimum results would be actually somewhat more than shown. In all other cases, however, it is believed that every result plotted on the diagrams refers to a single experiment.

* "On the Practical Strength of Columns, and of Braced Struts," *Minutes of Proceedings*, Institution of Civil Engineers, Vol. lxxxvi (1886).

As far as the writer is aware, Hodgkinson's tests of columns of Low Moor, No. 3, cast iron and of wrought iron, with both ends rounded or both ends flat, are shown here complete and in their full number (within the limits of length of the diagrams) for the first time.

Hodgkinson's other experiments on cast-iron columns are also shown on a diagram to which further reference will be made, but these tests were on specimens of various kinds of cast iron, and by themselves could form no reliable guide, although, together with Hodgkinson's experiments on Low Moor, No. 3, cast iron, they have formed practically the only basis for the design of cast-iron columns for the last fifty years; and extensive tables of the strength of cast-iron columns, based on them, and calculated from Gordon's or Rankin's formulas, are to be found in nearly every pocketbook published for the use of engineers.

The diagrams have all been plotted to such proportions as to show clearly the differences between the various tests and also their relation to the calculated curves derived from the writer's formulas. It would have been easy to have shown an apparently better agreement by adopting other proportions for the diagrams.

The writer proposes to deal with the theory and resulting formulas in the first place, and afterward to compare them with the results of nearly all the more important series of tests hitherto made upon columns of cast iron, wrought iron, steel and timber.

Theoretical Principles.

Any column with the smallest eccentricity of loading or the smallest amount of initial curvature will immediately begin to deflect under load, and the deflection will increase in a much more rapid degree than the increase of load, and every increase in deflection tends still further to increase deflection. It is this last fact which makes a column of moderate length so exceedingly sensitive to small deviations in loading or to small initial bends.

In order to arrive at an expression for the strength of any column, it is therefore necessary to develop first a formula to express the deflection of that column under given conditions. The deflection of a column is caused solely by the bending moments imposed upon it, and the laws of deflection are therefore the same as for a beam under transverse stress.

The probable deflection of a solid beam subjected to bending moments can be determined very simply, with all necessary practical accuracy, if the relations existing between the stress diagram and the resulting deflection are made use of.*

These relations apply with equal correctness to the case of a column. Let $P\ Q$ in Fig. 2 represent a cantilever of uniform section, and let $P\ Q\ R$ be the diagram of bending moments—either regular or irregular in outline. Let $I =$ moment of inertia of the cross-section of the cantilever, and $E =$ modulus of elasticity of the material used.

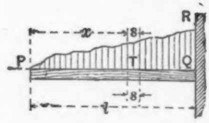


FIG. 2.

Take sections at distances x and $x + s$ from the point P at the extreme outer end of the cantilever, and let it be required to find the deflection of the point P below the point T at the center of s, caused by the stress existing between the two sections, which are further supposed to be exceedingly close together, so that s is very small compared to x. This being so, the bending moment on the cantilever between the two sections may, without appreciable error, be assumed as uniform for the length s.

Then, if M be the bending moment at the sections considered, and c, c' the distances of the extreme fibers from the neutral axis of the sections, the resulting stress in the fibers will be

$$f = \frac{Mc}{I} \text{ and } f' = \frac{Mc'}{I},$$

the common formula for the stress in a solid beam.

The resulting extension—or compression—of these extreme fibers will be $\lambda = \frac{fs}{E}$ and $\lambda' = \frac{f's}{E}$, and the point P will be deflected below the point T, as shown in Fig. 3 by dotted lines, and it is clear that, if we reduce s to an exceedingly small quantity, $\frac{\delta}{x} = \frac{\lambda}{c} = \frac{\lambda'}{c'}$, since the angle subtended by δ is very small; and therefore

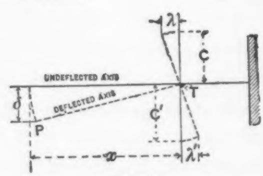


FIG. 3.

$$\delta = \frac{\lambda x}{c} = \frac{fsx}{Ec} = \frac{Mcsx}{IEc} = \frac{Msx}{IE}.$$

*"Theory of Solid and Braced Elastic Arches," by W. Cain, M. Am. Soc. C. E., 1879, and "Continuous-Girder Bridges," by T. C. Fidler, *Minutes of Proceedings*, Institution of Civil Engineers, Vol. lxxiv, 1883.

We have here dealt only with the deflections resulting from the bending moment and stress existing between the two sections, at distances x and $x + s$ from the point P, but the same relation holds for all other sections, and the total deflection caused by the bending moments along the whole length l of the cantilever will be $\Sigma\ \delta = \varDelta =$ the sum of the values of $\frac{M s x}{E I}$ between the points P and $Q = \Sigma \frac{M s x}{E I}$.

The numerator of this quantity is simply the moment of the area of the diagram of bending moments around the extreme point P, and the equation may be put into the form

$$\varDelta = \frac{A X}{E I},$$

where A = area of diagram of bending moments;

X = distance of center of gravity of this area from the point P.

It is to be remembered that the deflection to be dealt with in beams is always practically very small, compared to the length of the beam, and the above reasoning is not intended to be applied to absurd and improbable extremes. The relations deduced would not, of course, apply to a girder with a deep braced web under heavy shearing stresses, but on a solid beam section the deflection due to shearing stress is very small compared to that due to transverse bending; and, in the case of columns, whether solid or with braced webs, the shearing stresses are again much less than are usual in beams.

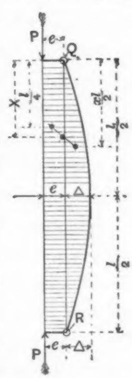


FIG. 4.

It is a simple matter to apply the foregoing relations to the case of a column under eccentric load, and with round or perfectly free pivoted ends. Let $Q R$ (Fig. 4) represent the axis of a column of length l acted upon by forces P acting at a distance e from the axis at its ends, with a resulting deflection $\varDelta$ at the center of the column's length.

The column deflection is exaggerated in the figure, for the sake of clearness, and the length l marked thereon appears to be considerably shorter than the true length of the curved axis of the bent column, but the difference in an actual column under test would be very small, and would not affect the reasoning to any appreciable extent.

The diagram of bending moments will then be as shown in the

figure included between the curve of the bent column and the line of action of the end forces $P\,P$, and, treating each half of the column as a cantilever, the deflection from the tangent to the bent column at its center will be $\Delta = \frac{A\,X}{E\,I}$ when equilibrium is established between th bending moments produced by the eccentric load with the resulting deflection, and the internal moment of resistance of the column sections.

In order to solve this equation for any given case, it is necessary, theoretically, to know the exact character of the curve of the bent column, as the bending moment at any point $= P\ (e + \Delta')$, where Δ' is the deflection from the chord line at that point, but we are mainly concerned with the deflection and bending moment at the point where these are maxima, *i. e.*, at the center of the column length, and the central deflection is the very quantity that is sought.

Practically, however, it is of small importance to know the precise nature of the curve, and a considerable divergence from theoretic accuracy in our knowledge of its true character makes but a trifling difference to the final result, as may be proved easily by assuming various outlines for the curve.

If the bending moment were perfectly uniform for the full length of the column, the curve taken by the latter would be part of a circle, and if the eccentricity of loading were indefinitely small, the curve would be a curve of sines. The actual curve is somewhere between these, and depends on the amount of eccentricity, and, as the deflection of columns in actual test or practice is very small in comparison with the column length, it is sufficiently accurate for all practical purposes to assume the curve to be a parabola, which will, under actual conditions, differ in an exceedingly small degree from the curves of both of the extreme conditions mentioned.

Under any circumstances, the area of the diagram of bending moments for the half length of a column, loaded as shown in Fig. 4, will be

$$A = \left(P \times e \times \frac{l}{2}\right) + \left(P \times \frac{\Delta\,y\,l}{2}\right),$$

when y is a coefficient expressing the mean deflection in terms of the maximum deflection Δ at the center of the column length, and the moment of this area around the extreme end Q of the column will be

$$A\,X = \overline{\left(P \times e \times \frac{l}{2}\right)\frac{l}{4}} + \overline{\left(P \times \frac{\Delta\,y\,l}{2}\right) \times \frac{x\,l}{2}},$$

where x is a coefficient in terms of $\frac{l}{2}$, expressing the distance from Q of the center of gravity of the portion of the moment diagram included between the chord and the curve of the bent column, *i. e.*, that portion of the moment diagram due to column deflection.

Then
$$A\,X = \frac{P\,l^2\,e}{8} + \frac{P\,l^2\,\varDelta\,y\,x}{4}$$
$$= \frac{P\,l^2}{8}(e + 2\,\varDelta\,y\,x),$$

and
$$\frac{A\;X}{E\,I} = \varDelta = \frac{P\,l^2}{8\,E\,I}(e + 2\,\varDelta\,y\,x);$$

and therefore
$$8\,E\,I\,\varDelta = P\,l^2\,(e + 2\,\varDelta\,y\,x),$$

from which
$$\varDelta = \frac{P\,l^2\,e}{8\,E\,I - 2\,P\,l^2\,y\,x} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now, making use of the assumption that the curve of the bent column is a parabola, the corresponding values of y and x will be

$$y = \frac{2}{3},\ \text{and}\ x = \frac{5}{8};$$

and substituting these values in General Equation (1), we have as a practical result for the deflection of columns under eccentric load

$$\varDelta = \frac{P\,l^2\,e}{8\,E\,I - (2\,P\,l^2 \times \frac{2}{3} \times \frac{5}{8})} = \frac{P\,l^2\,e}{8\,E\,I - \frac{5}{6}\,P\,l^2} \ldots\ldots \quad (2)$$

An inspection of these formulas at once shows that any column, of whatever material, with both ends round, and with the eccentricity of loading reduced to an exceedingly small degree, in fact, to as small an amount as we can form any conception of, so long as it has a positive value, would tend to have an infinite deflection, and therefore fail absolutely, as soon as the denominator of the right-hand member of the equation becomes zero.

Then, using Equation (1), $\varDelta$ would have an infinite value when

$$8\,E\,I = 2\,P\,l^2\,y\,x;$$

and the ultimate load would therefore be

$$P = \frac{8\,E\,I}{2\,l^2\,y\,x}.$$

Under the assumed ideal condition of perfect central loading, the curve of the column when bent being a curve of sines, the values y

and x would each be $\frac{2}{\pi}$, and substituting these in the equation, we have as the ultimate load of the ideal column with both ends round

$$P = \frac{\pi^2 E I}{l^2} = \frac{9.87 E I}{l^2}$$

or Euler's formula.

If Equation (2), based on the curve of the column being assumed to be a parabola, were used in this way to estimate the ultimate load, instead of General Equation (1), with the correct values of y and x, the result would be

$$P = \frac{9.6 E I}{l^2},$$

so that by applying the assumption as to the curve of the column being a parabola, even to the ideal extreme, we would make an error on the safe side of only 2.73% below the theoretic truth in the estimate of ultimate load, and a small amount of eccentricity, such as we may reasonably expect in presumably centrally-loaded columns in actual practice, would reduce this already small error to still smaller and practically inappreciable dimensions.

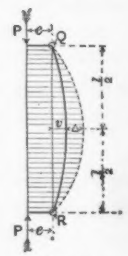


FIG. 5.

It is clear, therefore, from the foregoing, that the formulas for deflection, whether in the general form (1), or in the suggested practical form (2), are based on correct theoretic principles, and are of correct form.

We have dealt hitherto with the case of a column supposed to be absolutely straight before loading, and it remains to be seen what influence a small amount of initial curvature would have on the deflection.

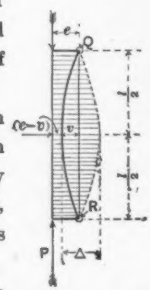


FIG. 6.

Let v, Figs. 5 and 6, represent the versed sine of an initial curvature, whether outwardly visible or not, in the axis of an eccentrically loaded column. It may have positive value, Fig. 5, or negative value, Fig. 6, relatively to the eccentricity e with which the load is imposed.

Let y_1 and x_1 be functions with regard to the area enclosed between the initial curve of the column and its chord line, of similar character to y and x, already adopted with regard to the curve of the column resulting from stress, or let y_1 and x_1 bear similar relations to v and l as y and x bear to Δ and l.

Then, whatever be the precise character of the initial curve and the deflection curve, using the same reasoning as before, we have

$$\frac{A\ X}{E\ I} = \frac{P\ l^2\ (e + 2\ y\ x\ \Delta \pm 2\ y_1\ x_1\ v)}{8\ E\ I} = \Delta$$

= central deflection due to stress,

and therefore $\Delta = \dfrac{P\ l^2\ (e \pm 2\ y_1\ x_1\ v)}{8\ E\ I - 2\ P\ l^2\ y\ x}$ .... (3)

This would give a minus value to Δ, if the quantity $2\ y_1\ x_1\ v$ should happen to have the minus sign, and at the same time be greater in value than e, but this would simply mean that the deflection would take place in the opposite direction to that in which e alone would cause it to bend. It must be kept in view that we are dealing with small amounts of initial curvature, shown in exaggeration in Figs. 5 and 6, for the sake of clearness.

In the case of Fig. 5, where the initial curvature is positive with regard to the eccentricity, *i. e.*, acts with the eccentricity to increase the deflection, the diagram of bending moments increases with the load imposed from the area bounded by the straight line joining the end forces $P\ P$, and the initial curve of the column, to the area bounded by the line joining $P\ P$ and the curve of the deflected column.

The bending moment at the center of the column length is primarily $P\ (e + v)$, increasing, as the column deflects under load, to $P\ (e + v + \Delta)$. In the case of Fig. 6, the conditions are altered by the initial curvature acting against the eccentricity, and the primary bending moment at the center of the column length will be $P\ (e - v)$, increasing to $P\ (e - v + \Delta)$, or $P\ (e - v - \Delta)$, depending on whether the influence of e or v happens to be greater in producing the deflection Δ, since Δ must of necessity take the sign of the more influential of these quantities.

Now comparing Formula (3) with Formula (1), it will be seen that an initial curvature has an influence similar to an equivalent value of eccentricity of loading, and the term $(e \pm 2\,y_1\,x_1\,v)$ may be replaced by a single term represented by ε in those columns for which we cannot determine precisely the correct value e and v, such as the presumably centrally loaded column of ordinary material.

Substituting the symbol ε for e in Formula (2), we then have

$$\Delta = \frac{P\ l^2\ \varepsilon}{8\ E\ I - \frac{5}{6}\ P\ l^2} \quad \text{.........................(4)}$$

for the deflection of practical columns apparently straight, and apparently centrally loaded.

It must at once be recognized that it is practically impossible to assign any value for ε beforehand, for any particular column, for reasons already given.

The foregoing reasoning will explain how it may easily happen that two columns of identical dimensions and of identical material, as far as we are able to determine, might give very different results in the testing machine, by one having the internal v acting with the accidental e, and the other having its internal v acting contrary to its accidental e, since the value of e might be very appreciable in the first case, and in the second case, if $-2\,y_1\,x_1\,v$ happened to be equal to $+\,e$, the value of ε would be zero, and the column would probably show a high ultimate strength.

We are now in a position to deduce a formula to express the maximum stress in a column under eccentric load, and to extend it to the case of the practical column under presumably central load.

Referring to Fig. 4, the bending moment at the center of the column length is

$$M = P\,(e + \Delta) = \frac{I f_b}{c} = \frac{a\,r^2 f_b}{c},$$

and therefore $$f_b = \pm\,\frac{P\,(e + \Delta)\,c}{a\,r^2},$$

where r is the radius of gyration of the column section in the direction in which the column bends, and f_b represents the unit stress caused by the bending moment alone at a distance c from the neutral axis, and a = the sectional area of the column.

The direct compression on the column section at the same time is $+f_a = \dfrac{P}{a}$ = average load per square inch on the sectional area of the column, and the total stress in the extreme fibers will therefore be

$$F = \pm f_b + f_a = \pm\,\frac{P\,(e + \Delta)\,c}{a\,r^2} + \frac{P}{a},$$

and now, substituting the value of Δ from Equation (2),

$$F = \frac{P}{a} \pm \left(\frac{P\left(e + \dfrac{P\,l^2\,e}{8\,E\,I - \frac{5}{6}\,P\,l^2}\right) c}{a\,r^2} \right) =$$

$$\frac{P}{a}\left\{ 1 \pm \frac{c}{r^2}\left(\frac{P\,l^2\,e}{8\,E\,I - \frac{5}{6}\,P\,l^2} + e \right) \right\}$$

$$= f_a \left\{ 1 \pm \frac{c\,e}{r^2}\left(\frac{f_a\,l^2}{8\,E\,r^2 - \frac{5}{6}\,f_a\,l^2} + 1 \right) \right\} \dots\dots\dots\dots\dots (5)$$

From Equation (5) it is probable that an expression may be deduced to give the average load per square inch f_a corresponding to a given maximum or minimum stress F with varying values of the other factors, but the writer has found it much simpler and easier to deal with the value of the ratio $\frac{l}{r}$ corresponding to a given value of maximum or minimum stress F, average stress f_a, modulus of elasticity E, and a given value of $\frac{c\,e}{r^2}$.

Efforts have been made, in connection with most column formulas, to determine the value of f_a for a given value of $\frac{l}{r}$, but the writer has never found, in his own practice, any advantage in this, and it is equally convenient to be able to determine the value of $\frac{l}{r}$ corresponding to a given value of f_a. Either way is, as a matter of practice, equally suitable for the purpose of laying down a curve to express the strength of varying proportions of columns.

From Equation (5), the general expression for the maximum stress produced in a column, we have

$$F = f_a \left\{ 1 \pm \frac{c\,e}{r^2} \left(\frac{f_a\,l^2}{8\,E\,r^2 - \frac{5}{6} f_a\,l^2} + 1 \right) \right\}$$

and, dividing the factor $\frac{f_a\,l^2}{8\,E\,r^2 - \frac{5}{6} f_a\,l^2}$ by r^2,

$$F = f_a \left\{ 1 \pm \frac{c\,e}{r^2} \left(\frac{f_a \frac{l^2}{r^2}}{8\,E - \frac{5}{6} f_a \frac{l^2}{r^2}} + 1 \right) \right\},$$

and using the symbol R to represent $\frac{l}{r}$, we have

$$F = f_a \left\{ 1 \pm \frac{c\,e}{r^2} \left(\frac{f_a\,R^2}{8\,E - \frac{5}{6} f_a\,R^2} + 1 \right) \right\}$$

$$= f_a \left\{ 1 \pm \frac{c\,e}{r^2} \left(\frac{f_a\,R^2 + 8\,E - \frac{5}{6} f_a\,R^2}{8\,E - \frac{5}{6} f_a\,R^2} \right) \right\}$$

$$= f_a \left\{ 1 \pm \frac{c\,e}{r^2} \left(\frac{48\,E + f_a\,R^2}{48\,E - 5\,f_a R^2} \right) \right\} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

and now using the + sign in the brackets to determine the maximum fiber stress F_c

$$F_c = f_a \left(1 + \frac{c\,e}{r^2} \left\{ \frac{48\,E + f_a\,R^2}{48\,E - 5\,f_a\,R^2} \right\} \right)$$

from which we have, by simple algebraic transformation, the value of R corresponding to any fixed value of the maximum compressive stress F_c,

$$\text{or } R = \sqrt{\frac{48\,E}{5\,F_c + f_d\left(\frac{c\,e}{r^2} - 5\right)}\left[\frac{F_c}{f_d} - 1 - \frac{c\,e}{r^2}\right]} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Similarly, using the — sign in the brackets in Equation (6) to determine the minimum fiber stress (not necessarily tensile) F_t.

$$F_t = f_d\left\{1 - \frac{c\,e}{r^2}\left(\frac{48\,E + f_d\,R^2}{48\,E - 5\,f_d\,R^2}\right)\right\}$$

which is easily transformed to

$$R = \sqrt{\frac{48\,E}{5\,F_t - f_d\left(\frac{c\,e}{r^2} + 5\right)}\left[\frac{F_t}{f_d} - 1 + \frac{c\,e}{r^2}\right]} \quad\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

It should be noted here that F_t only becomes tensile when the maximum tensile fiber stress f_b caused by bending is greater than the direct compressive stress f_d, and it should also be noted that F_t must be given its proper sign to correspond with its character, *i. e.*, + when compressive and — when tensile, irrespective of the fixed signs shown in Equation (8).

The precise use of these equations (7) and (8) is as follows:

Let it be assumed that in a given section of column, we decide that a certain value of maximum compressive stress F_c, or a certain value of minimum stress F_t, is not to be exceeded; these values being inserted in Formulas (7) and (8), together with the value of E and $\frac{c\,e}{r^2}$, corresponding to the material used and the section of column and eccentricity of loading actually adopted, we have at once the value of R corresponding to different values of f_d, the direct load per square inch.

Both of these formulas reduce to exceedingly simple terms on the insertion of the physical constants E, F_c, or F_t, and the proper value of $\frac{c\,e}{r^2}$, as will be seen later in their application.

Formulas (4), (5), (6), (7) and (8) are all general expressions applicable to the case of columns with both ends free, and of any given material and form of section, and with any given value of eccentricity of loading probable in practical work.

Professor William Cain* came to the conclusion that, with an ideal column, perfectly centrally loaded, up to the value given by Euler's

* "Theory of the Ideal Column," *Transactions*, Am. Soc. C. E., Vol. xxxix.

formula, a very small increase to this load insures failure of the column. A very similar conclusion can be drawn by applying Formula (4) to a given example, say $E = 30\,000\,000$, $I = 10$ ins.^{4}, $l = 300$ ins., and assume $e = 0.001$ in., corresponding to an accuracy of loading far beyond practical possibilities.

$$\text{Then } \Delta = \frac{Pl^2 e}{8\,EI - \frac{5}{6}Pl^2} = \frac{P \times 90\,000 \times \frac{1}{1000}}{2\,400\,000\,000 - 75\,000\,P}$$

and if P be taken in units of 10 000 lbs., this reduces to

$$\Delta = \frac{12\,P}{32\,000 - 10\,000\,P}.$$

Working this out for the various values of P, we have the results shown in Table No. 1.

TABLE No. 1.

P, in pounds.	Δ, in inches.	P, in pounds.	Δ, in inches.	P, in pounds.	Δ, in inches.
0	0.0000	31 600	0.0948	31 970	1.2788
20 000	0.002	31 650	0.1085	31 975	1.5348
25 000	0.0043	31 700	0.1268	31 980	1.9188
28 000	0.0084	31 750	0.1524	31 985	2.5588
29 000	0.0116	31 800	0.1908	31 990	3.8388
30 000	0.018	31 850	0.2548	31 992	4.7988
30 500	0.0244	31 900	0.3828	31 994	6.3988
31 000	0.0372	31 950	0.7668	31 996	9.599
31 500	0.0756	31 960	0.9588	32 000	∞

These results are instructive, and it hardly needs a calculation of maximum fiber stress to show how great is the effect of the small additions near to the ultimate load of 32 000 lbs.

The value of I assumed $= 10$ ins.$^4 = a\,r^2$, and nearly corresponds to a rectangular solid section 3.307 ins. square, with area nearly 10.94 sq. ins., and least radius of gyration $= 0.955$ in. nearly, so that the ratio $\frac{l}{r} = \frac{300}{0.955} = 314$.

The maximum compressive fiber stress

$$F_c = \frac{P\,(e + \Delta)\,c}{a\,r^2} + \frac{P}{a} = \frac{P\,(0.001 + \Delta) \times 1.6535}{10} + \frac{P}{10.94 \text{ sq. ins.}}$$

from which we have, when

$P =$	$\Delta + e =$	F lbs. per square inch $=$
31 990	3.8398	23 200
31 992	4.7998	28 300
31 994	6.3998	37 000
31 996	9.6	53 600
32 000	∞	∞

A most interesting feature of these figures for maximum fiber stress is the theoretic assurance which they give as to the capacity of long columns to resist fatigue, even when loaded nearly up to the crippling point; and if the material of the column dealt with in the example be assumed to have a compressive elastic limit of, say, 40 000 lbs. per square inch, it will be seen that the column would be quite uninjured by an infinite number of loadings within 10 lbs. of its ultimate supporting power. This, of course, would only hold good if the load were imposed without the slightest dynamic effect, or impact.

This completes the investigation of what may be termed "the elementary column," of which, columns with fixed ends, flat ends and pin ends, may be considered as merely modifications.

It will be noticed that the formulas in each case include a term $\frac{c}{r}$ dependent on the form of column section, and the writer at one time hoped to find practical verification of the influence of form of section in the published results of experiments, but the influence of other factors is too great and the number of tests on any one form of section is too small to enable this to be done as yet.

Again, the number of experiments carried out with a value of eccentricity sufficiently great to make it a paramount factor is very small, and in the great mass of tests hitherto made, the endeavor has been to impose the load "centrally"; we must, therefore, substitute ε (see page 347) for e in the formulas when applying them to experiments under presumably central loads. Under these circumstances, it appears justifiable, in our present state of knowledge, to consider the factor $\frac{c\,\varepsilon}{r^2}$ as a constant, of which the value for centrally loaded columns must be determined from available experimental records.

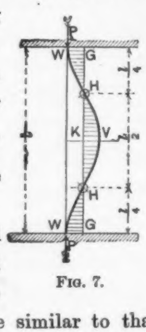


FIG. 7.

Attention will be given, next, to the fixed-ended column shown in Fig. 7, the section being assumed to be uniform, as before. When under load the column $W\,W$ will bend in a double reverse curve similar to that shown in the figure, and the central portion of the column $H\,H$ will behave similarly to, and be subject to, the same laws as a free-ended column. The question to be solved, in the first place, is as to the proportion of the total length of a fixed-ended column, which will act

as a free-ended column. This proportion is frequently stated to be one-half, without any reasons being stated.

In determining this it is convenient to consider one-half of the column only (since the other will necessarily behave in a precisely similar manner) as shown in Fig. 8, to a larger scale.

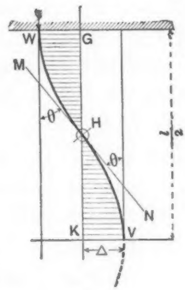


FIG. 8.

Referring again to Fig. 3 and its descriptive context it was shown that

$$\delta = \frac{M\ s\ x}{E\ I}$$

and therefore $\frac{\delta}{x} = \frac{M s}{E\ I}$ = the tangent of the angle of slope set up at the end of the length x by the stresses in the portion s of the cantilever considered; and as this angle of slope is in practice exceedingly small, its tangent will practically represent the angle in circular measure with all necessary accuracy, and the sum of all the exceedingly small angles of slope, for the full length of the cantilever, will then be

$$\Sigma\ \frac{\delta}{x} = \Sigma \frac{M s}{EI} = \frac{A}{E\ I} = \frac{\Delta}{X} = \Theta,$$

or the angle of slope at the extreme end of the cantilever is proportional to the area of the curve of bending moments.

As before, this reasoning and its results apply equally well to the bent column.

In Fig. 8 the tangent $M\ N$ to the curve of the bent column at the point of contrary flexure H is common to both portions of the curve, and the slope of each portion is, therefore, the same at this point, since the tangents at W and V remain vertical and parallel to each other, in consequence of the fixity of the end W and the symmetry of the whole column around the point V. The area of the bending moment diagram $G\ H\ W$ must, therefore, be equal to the area of the bending moment diagram $K\ H\ V$. Further, as the column section is assumed to be uniform, and there is no bending moment at the point H, where the two portions $W\ H$ and $H\ V$ react upon each other in simple compression and shear, the curvature of the two portions at corresponding points on either side of H must evidently be identical, inasmuch as both are of the same section, subject to the same forces and subject to the same laws of flexure.

From these conditions of equal curvature and equal area of bending moment diagram, it results that the length $G H$ must be equal to the length $K H$, and each of them will, therefore, be one-fourth of the total length of the column. Also the portion $W G$ of the total deflection must equal the portion $K V$, and the bending moment at the fixed ends W will equal the bending moment at the center V.

It is thus determined that the length of $H H$, Fig. 7, acting as a free-ended column, is one-half the total length L of the fixed-ended column, or, in other words, a fixed-ended column carrying a given load is twice the length of a free-ended column of the same section and having similar stresses.

This result is based on assumptions of perfect straightness before bending, perfectly homogeneous material, and perfect fixity of ends. In practical work some divergence will undoubtedly occur, which will require to be allowed for by an assumed equivalent eccentricity of loading, as in the case of the simple free-ended column already dealt with, and we therefore have, for the fixed-ended column:

$$R = 2\sqrt{\frac{48\,E}{5\,F_c + f_d\left(\frac{c\,\varepsilon}{r^2} - 5\right)}\left[\frac{F_c}{f_d} - 1 - \frac{c\,\varepsilon}{r^2}\right]} \quad \text{for failure by compression,}$$

$$\text{or } 2\sqrt{\frac{48\,E}{5\,F_t - f_d\left(\frac{c\,\varepsilon}{r^2} + 5\right)}\left[\frac{F_t}{f_d} - 1 + \frac{c\,\varepsilon}{r^2}\right]} \quad \text{for failure by tension.}$$

In actual practice the true fixed-ended column rarely, if ever, exists. It is difficult, even in experiments in a testing machine, to comply with the conditions necessary to ensure absolute fixity of ends, and in ordinary construction the difficulty is increased greatly.

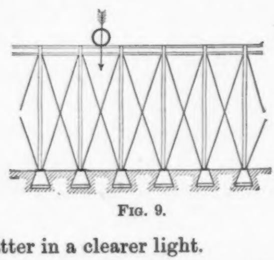


FIG. 9.

The writer has found that the vaguest ideas are sometimes held as to what is required to realize fixed ends in a column. A consideration of simple examples will probably exhibit this matter in a clearer light.

Assume, in the first instance, that we have, as in Fig. 9, a series of stiff gantry girders, 2 ft. deep by 10 ft. span, riveted securely to the heads of columns 30 ft. high, firmly braced together to preserve their

verticality. Assume, also, that the foundation blocks on which the column rests are very rigid, that the columns have large well-bolted bases, and that the ratio $\frac{l}{r}$ of these columns is very large, and the columns therefore slender in proportion.

Then the imposition of load on any span will cause deflection in the girder, and the ends of the girder will deviate from the vertical to a slight degree, but the relative stiffness of the girders themselves, as compared with the column, being high, the approximation to ideal fixity of ends would, practically speaking, be of a high degree.

In the second instance, Fig. 10, let the columns be spaced at 30 ft. centers, retaining the same depth of girder, 2 ft., and merely increasing the girder sections to obtain the same value of working unit stress, while increasing the radius of gyration of the columns to provide much greater stiffness of column. Under these conditions, the deflection of the girder under load, and consequently the slope of the ends of the girders where they are securely riveted to the column heads, would be increased largely, and the columns would be subjected to heavy bending stresses in addition to their direct load. These columns would be much less heavily stressed if they had pin-joint connections to the girders, and the apparent fixity of end, given by a secure riveted connection, would actually be accompanied by severely prejudicial secondary stresses.

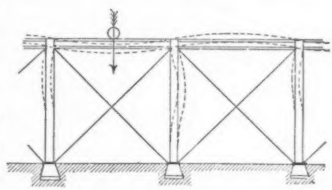


FIG. 10.

The conclusion derived from these examples is that in practical work the degree of approximation to fixity of ends depends entirely on the relative stiffness of the column and the other members of the structure attached to it; and the estimation of this degree of fixity demands the most careful consideration on the part of the engineer.

One of the advantages claimed for riveted connections in bridge work is that the compression members are thereby made into fixed-ended columns, and can be accorded higher stresses in consequence.

A portion of a riveted main girder of N-type is shown in Fig. 11, with connections of web members for two panel points, and the writer would ask whether the top boom fixes the ends of the vertical posts, or do the posts fix the ends of the panel lengths in the top boom (which is a column between panel points), or are we to rely upon the stiffness of the diagonal tension members to fix both?

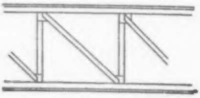


FIG. 11.

The last can hardly be considered a reasonable assumption, as the most heavily loaded portions of the top boom are at the center of the span, where also the lightest diagonals are found, and as regards the posts fixing the ends of the panel lengths of the top boom, this also is out of the question, as the stiffness of the posts is usually small as compared with the stiffness of the boom, and if we consider the top boom as fixing the ends of the vertical posts, under which class of columns are we to place the top boom panel lengths? They could not be considered as fixed-ended, and as they would have to perform the additional duty of fixing the vertical post ends, they could not be considered to be as favorably circumstanced as a round-ended or pivot-ended column. Here, again, we have to give consideration to relative stiffness of parts.

It is common knowledge that heavy secondary stresses exist in the connections of various members in a riveted structure, but it is not commonly recognized that these very secondary stresses may totally destroy any imaginary fixity of ends in the compression members, and actually place the members under worse conditions of stress than if pivoted end-bearings were adopted.

The writer is neither seeking to depreciate the practical value of the riveted connections nor to advocate either pin or pivoted end connections, but only wishes to point out the erroneous principles on which designs are frequently based.

As far as the writer is aware, no attempt has hitherto been made to arrive at a rational basis for the strength of flat-ended columns, although the greater number of tests of columns have been made with this class of end-bearing. As a rule, the assumption has been made that they act in precisely the same manner as fixed-ended columns, and column formulas to cover both in one expression are frequently given. This is quite erroneous, both from a theoretical point of view,

and from the evidence of actual experiments. With flat ends, no tensile stress can be developed at the ends, and with fixed ends it has been shown that the bending moment at each end is theoretically equal to that at the center of the column.

It is clear, then, that so long as no tensile stress is set up in a column with flat ends, it will behave as a fixed-ended column, that is, up to the point of loading at which the stress in the column at its ends and center is as shown in Fig. 12, when a small increase of load on the column will most probably cause failure.

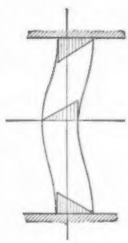


FIG. 12.

If, then, we use the formula for minimum stress F_t, and make F_t equal to zero, we obtain the value of R at which a given value of f_d will produce a minimum stress zero, and we will thus determine the value of R corresponding to incipient tensile stress, and the formula with F_t inserted as of zero value, will give what may be called the critical value,

$$R = 2\sqrt{\frac{48\,E}{0 - f_d\left(\frac{c\,\varepsilon}{r^2} + 5\right)}\left(0 - 1 + \frac{c\,\varepsilon}{r^2}\right)}$$

$$= 2\sqrt{\frac{48\,E\left(-1 + \frac{c\,\varepsilon}{r^2}\right)}{-f_d\left(\frac{c\,\varepsilon}{r^2} + 5\right)}}$$

$$= 2\sqrt{\frac{48\,E\left(1 - \frac{c\,\varepsilon}{r^2}\right)}{f_d\left(\frac{c\,\varepsilon}{r^2} + 5\right)}} \quad \left\{\begin{array}{l}\text{when tension is incipi-}\\ \text{ent in flat-ended}\\ \text{columns.}\end{array}\right\} \quad \ldots (9)$$

Here, again, it is necessary to determine from experimental results what value must be given to the factor $\frac{c\,\varepsilon}{r^2}$, if the formula is to be used to determine ultimate strength.

It may be urged that the load producing incipient tension in any given flat-ended column may not be the ultimate load; but, as soon as tension is attempted to be set up at the ends of a flat-ended column, the column will be in a highly unstable condition, and the ends will begin to rotate on their bearing faces. This is most readily seen by considering the case of a fixed-ended column in which tensile stresses have been set up at the ends and center.

If the fibers in tension at the ends were cut, so as to transform the column into a flat-ended column while under load, we would naturally expect the column to alter its curvature immediately, and largely increase its deflection, with the result that it would probably fail immediately, or with a comparatively small additional load. The substantial truth of this, in practice, is most clearly evident in Mr. Christie's experiments, as will appear later.

By plotting the curves for the two conditions, one for failure by maximum compressive stress, and the other for the critical condition of incipient tension, it is made evident that with any given section of column, up to a certain value of R, dependent on the eccentricity of loading and modulus of elasticity, no tension can be set up in the column, whatever the load, and flat-ended columns below this limiting proportion behave in every sense as fixed-ended columns, while beyond this point the strength will fail more or less rapidly.

The writer believes that the value of the difference in the strength of fixed and flat-ended columns is here dealt with in a rational manner for the first time.

With regard to pin-ended columns, it is quite useless to theorize with the view of showing their superiority to round or pivot ends, owing to the fact that their behavior under load, even in a testing machine, depends very largely on the closeness of the fit between pin and hole, upon the smoothness or otherwise of the bearing surfaces, upon the diameter of the pin in relation to the radius of gyration, and upon the presence, either accidental or premeditated, of a lubricating medium.

In actual practice, the vibration in a railway bridge, caused by the passage of the load, and the movements of the members relatively to each other under the common variations of stress, must undoubtedly go very far to destroy the friction upon which depends the superiority with which this type of strut is often credited over those with round ends.

There is as yet no satisfactory and conclusive evidence that in practical work the pin-ended column can fairly be credited with this greater strength, and the practice of imposing higher stresses on account of the pin ends is open to grave question.

This matter may be viewed from another standpoint, that of the advantages claimed for the pin joint as compared with the riveted con-

nection. Among these so-called advantages are freedom from secondary stresses and greater certainty of realizing the ideal condition of centrality of loading on the various members.

Any additional strength accompanying the pin-bearing type of column can only be obtained when frictional resistances are set up in the bearing, preventing rotation, and thus bringing into play a moment of resistance to bending on the column end, and this moment of resistance in turn can only be developed by subjecting the other members assembled on the same pin to secondary bending stresses in order to realize a partial fixity of column ends.

In any case, the additional resistance due to partial fixity of ends in the pin-ended column, if it actually exists in practical construction, must be obtained at the expense of the other members on the same pin, and is largely dependent on the stiffness of those members. The question may fairly be raised whether or not it would be consistent practice to make an allowance for the secondary stresses in these other members, if we rely on these secondary stresses to provide the column with increased resistance.

Comparison of the Formulas with the Experiments.

Deflection Formula (4) for Round-Ended or Pivot-Ended Columns.

$$\Delta = \frac{P\,l^2\,\varepsilon}{8\,E\,I - \frac{5}{6}\,P\,l^2}.$$

The starting point for the whole of the foregoing theory and formulas was the development of this expression for the deflection of a column, and it has already been pointed out that, in practical work, some of the controlling influences cannot be made subject to actual observation, depending as they do on internal conditions arising from past history, accidental errors in setting, etc., etc.

Nevertheless, it is important to have some definite knowledge as to whether the deflection formula bears characteristic features having practical agreement with the results of actual observations in experiments.

In order to make a comparison it was necessary to make numerous trial calculations with different estimates of the values of those factors which are primarily unknown, and the process of comparison therefore consisted of fitting the calculated values given by the formula to the observed deflections, so that it might be seen whether curves

plotted with loads as abscissas and deflections as ordinates have the same character by calculation and observation.

The unknown factors for which the values have had to be estimated are the following:

1. The modulus of elasticity.................................. E,
2. The equivalent eccentricity............................... ε,
3. Any small amount of initial curvature capable of observation, V, but not always noted in records of test and making the total deviation of the column from a straight line $= D = \varDelta \pm V$.

It was found that the influence of each of these three factors is so great that very small deviations from the estimated values finally adopted destroy the agreement between calculation and observation, and as, in these estimated values, we are already dealing with very small quantities, it is to be noted that the small deviations referred to would be incapable of being observed in any ordinary experiment.

The examples selected for purposes of comparison have been taken from the tests of round-ended columns of Low Moor No. 3 cast iron, made by Mr. Eaton Hodgkinson,* and from the tests of round-ended wrought-iron columns, made by Mr. James Christie.†

These examples have been chosen solely on account of the fullness of the records of deflections. In each case the effective column length has been taken as being the distance between the centers of the hemispherical ends, as shown in Fig. 13.

FIG. 13.

Examples from Hodgkinson's Tests of Low Moor, No. 3 Cast Iron.

Test No. 5, of Table I.‡—Solid, cylindrical, hemispherical-ended column, 60.50 ins. length over all, 0.99 in. diameter, say 59.50 ins. effective length.

$$I = \text{moment of inertia of section} = 0.0472 \text{ in.}^4$$

Estimated values:
- $E = 14\,500\,000$ lbs., modulus of elasticity;
- $\varepsilon = 0.055$ in., equivalent eccentricity of loading;
- $V = +\,0.07$ in., probable initial curvature escaping observation.

* Recorded in the *Philosophical Transactions* of the Royal Society of London, for 1840.

† *Transactions*, Am. Soc. C. E., Vol. xiii.

‡ *Philosophical Transactions*, Royal Society, London, 1840.

	P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
	515....	0.0253	+ 0.07	= 0.0953..........	0.05
	655....	0.0360	+ 0.07	= 0.1060..........	0.10
	991....	0.0756	+ 0.07	= 0.1456..........	0.14
	1 183....	0.1160	+ 0.07	= 0.1860..........	0.19
	1 471....	0.2525	+ 0.07	= 0.3225..........	0.32
	1 615....	0.4425	+ 0.07	= 0.5125..........	0.52
Ult. load.	1 663....	0.5690	+ 0.07	= 0.6390..........	$\propto$ failed.

Test No. 20, of Table I.—Solid, cylindrical, hemispherical-ended column, 60.50 ins. length over all, 1.97 ins. diameter, say 58.50 ins. effective length.

$$I = 0.74 \text{ in.}^4$$

Estimated values: $E = 13\ 600\ 000$ lbs., $\varepsilon = 0.0675$ in., $V = 0$.

	P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
	3 355....	0.0109	+ 0	= 0.0109...........	bent.
	7 386....	0.0287	+ 0	= 0.0287...........	0.02
	12 970....	0.0686	+ 0	= 0.0686...........	0.07
	19 943....	0.1943	+ 0	= 0.1943...........	0.20
	21 035....	0.2360	+ 0	= 0.2360...........	0.23
	22 127....	0.2936	+ 0	= 0.2936...........	0.28
	23 219....	0.3740	+ 0	= 0.3740...........	0.37
	24 311....	0.5000	+ 0	= 0.5000...........	0.50 to 0.52
	24 857....	0.5940	+ 0	= 0.5940...........	0.60
Ult. load.	25 403....	0.7230	+ 0	= 0.7230...........	$\propto$ failed.
	(26 000)....	0.9350	+ 0	= 0.9350...........	....

Test No. 1, of Table VIII.—Hollow, cylindrical, hemispherical-ended column (Fig. 14), 90.75 ins., length over all, 1.78 ins. external diameter, 1.21 ins. internal diameter, say 89 ins. effective length.

Core center 0.19 in. out of center of external circle of column (ascertained after fracture). Center of area 0.163 in. out of center of external circle of column.

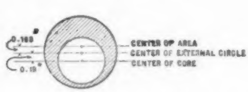


FIG. 14.

$$I = 0.31 \text{ in.}^4$$

Estimated values: $E = 17\ 500\ 000$ lbs., $\varepsilon = 0.1$ in., $V = -0.02$ in.

	P = load, in pounds.	(Δ Inches.	$-$ V) Inches.	= D calculated. Inches.	D observed. Inches.
	2 237....	0.06186	— 0.02	= 0.04186........	0.03
	2 813....	0.0897	— 0.02	= 0.0697.........	0.07
	3 317....	0.12208	— 0.02	= 0.10208........	0.11
	3 821....	0.1664	— 0.02	= 0.1464.........	0.16
	4 325....	0.23049	— 0.02	= 0.21049........	0.20
	4 829....	0.33153	— 0.02	= 0.31153........	0.32
	5 333....	0.51441	— 0.02	= 0.49441........	0.49
Maximum.	5 585....	0.67549	— 0.02	= 0.65549....	not observed; column not allowed to break.
	(6 000)....	1.2471	— 0.02	= 1.2271.....	

Test No. 5, of Table VIII.—Hollow, cylindrical, hemispherical-ended column (Fig. 15), 90.75 ins. length over all, 2.23 ins. external diameter, 1.53 ins. internal diameter, therefore, say 88.52 ins. effective length.

Core center was 0.135 in. out of center of external circle of column (ascertained after fracture). Therefore center of area was 0.12 in. out of center of external circle.

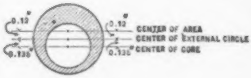


FIG. 15.

$$I = 0.8863 \text{ in.}^4$$

Estimated values: $E = 14\,500\,000$ lbs., $\varepsilon = 0.175$ in., $V = -0.025$ in.

	P = load, in pounds.	(Δ Inches.	$-$ V) Inches.	= D calculated. Inches.	D observed. Inches.
	2 237....	0.0347	— 0.025 =	0.0097.......	0.01
	4 325....	0.0796	— 0.025 =	0.0546.......	0.06
	6 341....	0.1415	— 0.025 =	0.1165.......	0.12
	8 357....	0.2375	— 0.025 =	0.2125.......	0.22
	9 365....	0.3085	— 0.025 =	0.2835.......	0.28
	10 373....	0.4040	— 0.025 =	0.3790.......	0.37
	11 381....	0.5475	— 0.025 =	0.5225.......	0.55
	12 137....	0.7080	— 0.025 =	0.6830.......	0.69
Maximum.	12 389....	0.7525	— 0.025 =	0.7275....	not observed; column not allowed to break.
	(13 000)....	(0.9950)	— 0.025 =	(0.9700)...	

Test No. 15, of Table VIII.—Hollow, cylindrical, hemispherical-ended column (Fig. 16), 90.75 ins. length over all, 3.36 ins. external diameter, 2.61 ins. internal diameter, say 87.39 ins. effective length $= l$.

Core center, after fracture, found to be 0.067 in. out of center of external circle of column. Therefore, center of area was 0.105 in. out of center of external circle.

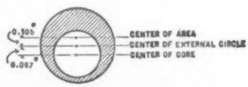


FIG. 16.

$$I = 3.915 \text{ ins.}^4$$

Estimated values: $E = 12\,000\,000$ lbs., $\varepsilon = 0.16$ in., $V = +0.035$ in.

P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
3 355	0.0116	+ 0.035	= 0.0466	bent.
16 115	0.0727	+ 0.035	= 0.1077	0.09
18 667	0.0895	+ 0.035	= 0.1245	0.13
21 729	0.1128	+ 0.035	= 0.1478	0.15
24 148	0.1341	+ 0.035	= 0.1691	0.17
28 986	0.1872	+ 0.035	= 0.2222	0.24
33 824	0.2625	+ 0.035	= 0.2975	0.30
37 701	0.3445	+ 0.035	= 0.3795	0.38
41 632	0.4680	+ 0.035	= 0.5030	0.48
43 597	0.5550	+ 0.035	= 0.5900	0.59
45 563	0.6650	+ 0.035	= 0.7000	0.67
47 528	0.8175	+ 0.035	= 0.8525	0.87
48 511	0.9140	+ 0.035	= 0.9490	0.90
49 494	1.0320	+ 0.035	= 1.0670	1.07
Ult. load. 50 477	1.1750	+ 0.035	= 1.2100	∝

Test No. 7, *of Table III.*—Solid, rectangular pillar with hemispherical ends, 60.5 ins. length over all, 1.54 x 1.56 ins. nearly square, say 59 ins., effective length $= l$.

$$I = 0.475 \text{ in.}^4$$

Estimated values: $E = 13\,250\,000$ lbs., $\varepsilon = 0.06$ in., $V = -0.015$ in.

P = load, in pounds.	(Δ Inches.	— V) Inches.	= D calculated. Inches.	D observed. Inches.
2 141	0.010	— 0.015	= — 0.005	column bent.
4 465	0.025	— 0.015	= + 0.010	0.015
6 481	0.043	— 0.015	= + 0.028	0.02
11 169	0.130	— 0.015	= + 0.115	0.11
13 565	0.257	— 0.015	= 0.242	0.23
14 461	0.360	— 0.015	= 0.345	0.35
14 909	0.438	— 0.015	= 0.423	0.44
15 357	0.552	— 0.015	= 0.537	0.52
Ult. load. 15 581	0.632	— 0.015	= 0.617	∝

Examples from Christie's Tests of Wrought-Iron Struts with Hemispherical Ends.*

Test No. 204.—T-bar; 1 in. x 1 in. x 99.25 ins. length over all, 1-in. balls and plates, 98.25 ins. effective length $= l$.

$$I \text{ (least)} = a\,r^2 = 0.3 \text{ sq. in.} \times (0.26)^2 = 0.02028 \text{ in.}^4$$

Estimated values: $E = 30\,000\,000$ lbs., $\varepsilon = 0.05$ in., $V = +$ 0.025 in.

	P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
	100	....0.0118	+ 0.025	= 0.0368........	0.05
	200	....0.0296	+ 0.025	= 0.0546........	0.05
	300	....0.0590	+ 0.025	= 0.0840........	0.08
	400	....0.1169	+ 0.025	= 0.1419........	0.15
	500	....0.2843	+ 0.025	= 0.3093........	0.30
Ult. load.	550	....0.5964	+ 0.025	= 0.6214........	∞

Test No. 205.—T-bar; 3 ins. $\times$ 3 ins. $\times$ 82.0625 ins. length over all, 2-in. balls and plates, 80.0625 ins., effective length.

$$I = a\,r^2 = 2.53 \text{ sq. ins. x } (0.62)^2 = 0.9725 \text{ in.}^4$$

Estimated values: $E = 25\,000\,000$ lbs., $\varepsilon = 0.02$ in., $V = 0.035$ in.

	P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
	500	....0.0003	+ 0.035	= 0.0353........	0.03
	5 000	....0.0038	+ 0.035	= 0.0388........	0.04
	10 000	....0.0091	+ 0.035	= 0.0441........	0.05
	15 000	....0.0168	+ 0.035	= 0.0518........	0.05
	20 000	....0.0293	+ 0.035	= 0.0643........	0.06
	25 000	....0.0526	+ 0.035	= 0.0876........	0.09
	30 000	....0.1122	+ 0.035	= 0.1472........	0.15
	34 000	....0.3380	+ 0.035	= 0.3730.....	not recorded;
Ult. load.	34 110	....0.3560	+ 0.035	= 0.3910.....	column failed.

* *Transactions*, Am. Soc. C. E., Vol. xiii.

Test No. 206.—T-bar; $2\frac{1}{2}$ ins. × $2\frac{1}{2}$ ins. × 82.375 ins. long over all, 2-in. balls and plates, 80.375 ins. effective length.

$$I = a\,r^2 = 1.73 \text{ sq. ins.} \times (0.55)^2 = 0.523325 \text{ in.}^4$$

Estimated values: $E = 32\,000\,000$ lbs., $\varepsilon = 0.035$ in., $V = +\,0.05$ in.

P = load, in pounds.	(Δ Inches.	+	V) Inches.	=	D calculated. Inches.	D observed. Inches.
500	0.00086	+	0.05	=	0.0508	0.03
3 000	0.0058	+	0.05	=	0.0558	0.05
6 000	0.0133	+	0.05	=	0.0633	0.06
9 000	0.0238	+	0.05	=	0.0738	0.07
12 000	0.0392	+	0.05	=	0.0892	0.10
15 000	0.0636	+	0.05	=	0.1136	0.12
18 000	0.1097	+	0.05	=	0.1597	0.16
20 000	0.1718	+	0.05	=	0.2218	0.22
21 000	0.2270	+	0.05	=	0.277	0.30
Ult. 21 500	0.2660	+	0.05	=	0.3160	∝ failed.
(22 000)	(0.3200	+	0.05)	=	(0.3700)	....
(23 000)	(0.5120	+	0.05)	=	(0.5620)	....

Test No. 208.—T-bar; $1\frac{1}{2}$ ins. × $1\frac{1}{2}$ ins. × 81.1875 ins. long over all, on 1-in. balls and plates, therefore, effective length = 80.1875 ins.

$$I = a\,r^2 = 0.53 \text{ sq. in.} \times (0.32)^2 = 0.054272 \text{ in.}^4.$$

Estimated values: $E = 30\,000\,000$ lbs., $\varepsilon = 0.05$ in., $V = 0.03$ in.

P = load, in pounds.	(Δ Inches.	+	V) Inches.	=	D calculated. Inches.	D observed. Inches.
200	0.00538	+	0.03	=	0.0354	0.02
600	0.01965	+	0.03	=	0.0497	0.05
800	0.29380	+	0.03	=	0.0594	0.06
1 000	0.04188	+	0.03	=	0.0719	0.07
1 200	0.05840	+	0.03	=	0.0884	0.09
1 400	0.08135	+	0.03	=	0.1114	0.12
1 600	0.11525	+	0.03	=	0.1453	0.15
1 800	0.17100	+	0.03	=	0.2010	0.20
2 000	0.27750	+	0.03	=	0.3075	0.30
Ult. 2 200	0.5696	+	0.03	=	0.5996	∞ failed.
(2 250)	(0.74000	+	0.03)	=	(0.7700)	....
(2 300)	(1.04200	+	0.03)	=	(1.0720)	....

Test No. 211.—T-bar; 2 ins. × 2 ins. x 63.1875 ins. long over all, on 1-in. balls and plates, therefore, effective length = 62.1875 ins.

$$I = a r^2 = (0.95 \text{ sq. in.}) \times (0.43)^2 = 0.175655 \text{ in.}^4$$

Estimated values: $E = 27\ 000\ 000$ lbs., $\varepsilon = 0.10$ in., $V = 0.02$ in.

	P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
	500	0.00533	+ 0.02	= 0.0253	0.03
	2 000	0.02455	+ 0.02	= 0.0446	0.05
	3 000	0.04110	+ 0.02	= 0.0611	0.06
	4 000	0.06175	+ 0.02	= 0.0818	0.08
	5 000	0.08860	+ 0.02	= 0.1086	0.10
	6 000	0.12460	+ 0.02	= 0.1446	0.15
	7 000	0.17600	+ 0.02	= 0.1960	0.20
	8 000	0.25450	+ 0.02	= 0.2745	0.27
	9 000	0.38900	+ 0.02	= 0.4090	0.40
	9 500	0.50100	+ 0.02	= 0.5210	not recorded.
Ult. load.	9 510	0.50400	+ 0.02	= 0.5240	$\propto$ failed.
	9 550	0.51600	+ 0.02	= 0.5360	....

Test No. 217.—T-bar; 1 in. × 1 in. × 45¼ ins., long over all, on 1-in. balls and plates, therefore, effective length = 44.25 ins.

$$I = a r^2 = 0.3 \text{ sq. in.} \times (0.26)^2 = 0.02028 \text{ in.}^4$$

Estimated values: $E = 18\ 500\ 000$ lbs., $\varepsilon = 0.037$ in., $V = 0.008$ in.

	P = load, in pounds.	(Δ Inches.	+ V) Inches.	= D calculated. Inches.	D observed. Inches.
	100	0.0025	+ 0.008	= 0.0106	0.01
	400	0.0123	+ 0.008	= 0.0203	0.02
	600	0.0215	+ 0.008	= 0.0295	0.03
	800	0.0342	+ 0 008	= 0.0422	0.04
	1 000	0.0533	+ 0.008	= 0.0613	0.06
	1 200	0.0832	+ 0.008	= 0 0912	0.10
	1 400	0.1415	+ 0.008	= 0.1495	0 15
	1 600	0.2968	+ 0.008	= 0.3048	0.29
	1 700	0.5410	+ 0.008	= 0.5490	0.55
Ult. load.	1 750	0.8675	+ 0.008	= 0.8755	$\propto$ failed.
	1 800	2.0250	+ 0.008	= 2.0330	....

With regard to the value of the modulus of elasticity E, as estimated for the foregoing comparative examples, it may be mentioned that Mr. Hodgkinson found from transverse bending tests of the Low

Moor No. 3 cast iron, that E ranged from 13 585 530 to 14 251 950 lbs., and Mr. James Christie* found from bending tests of the wrought iron upon which he experimented, that E ranged from 19 164 000 to 33 631 000 lbs.

The value of E, as found by bending tests, is necessarily that to which we must refer in dealing with column strength and stiffness, as the modulus of elasticity only enters into the column formula on account of the bending moments exerted on the column, and not at all in connection with direct compressive stresses.

It may be noted here that the value of E obtained from the transverse bending tests on ordinary cold-straightened wrought-iron or steel bars will depend upon the position of the points at which the straightening press has been applied. If the straightening is done near the center of the span, the value of E may reasonably be expected to come out very low, while the influence of any straightening done near the ends of the bars will have comparatively little influence on the results obtained.

In direct tensile tests the position of the points of straightening will have no influence on the results, which will only be affected by the amount of straightening to which the bar has been subjected. This will explain the much greater uniformity in the results obtained by direct tension, as compared with those obtained by transverse bending tests, and also with those obtained from compression tests where the slightest latitude is given for the specimen to act as a column, and where the material has had to be cold-straightened.†

General Formulas for the Relation of Column Proportions to Column Strength.

Round or pivoted ends—

Formula (7) $$R = \frac{l}{r} = \sqrt{\frac{48\,E}{5\,F_c + f_d\left(\frac{c\,\varepsilon}{r^2} - 5\right)}\left[\frac{F_c}{f_d} - 1 - \frac{c\,\varepsilon}{r^2}\right]}$$

for failure by compressive stress,

or (8) $$R = \frac{l}{r} = \sqrt{\frac{48\,E}{5\,F_t - f_d\left(\frac{c\,\varepsilon}{r^2} + 5\right)}\left[\frac{F_t}{f_d} - 1 + \frac{c\,\varepsilon}{r^2}\right]}$$

for failure by tensile stress.

* "The Strength and Elasticity of Structural Steel," *Transactions*, Am. Soc. C. E., Vol. xiii.

† See results of Mr. Christie's tests in "The Strength and Elasticity of Structural Steel," *Transactions*, Am. Soc. C. E., Vol. xiii.

Fixed ends.... R = twice the values given by Formulas (7) and (8).

Flat ends..... R = same as fixed ends, keeping in view that in Formula (8) F_t is to be made zero, resulting in Formula (9).

Hinged or Pin ends. } .. R = { in upper limits, same as for fixed ends. in lower limits, same as for round ends.

Attention must here be drawn to the fact that Formulas (7) and (8) are not actually two different formulas, but are only simple algebraic transformations of one and the same general formula (6), referred to the two conditions of failure by compressive stress and by tensile stress, the only other modifications necessary for their application to any of the types of column previously given being due to the conditions of end fixing as determining the relative length of columns of the same strength, but with different end conditions.

It will be noticed that there are three factors in the formulas to which it is necessary to assign values, and these are:

(1) The value of E = modulus of elasticity;

(2) The value of F_c or F_t = the maximum fiber stress;

(3) The value of $\frac{c\,\varepsilon}{r^2}$.

Careful study of the great variations shown in actual tests, and of the comparisons made between actual and calculated deflections, apparently indicate very great difficulty in assigning any fixed values for these quantities for general application, but it must be remembered, that it is not practically possible to predict the precise strength of any given column, and that being so, it only remains to endeavor to determine the limits between which we may expect the column strength to lie. With this in view, using approximately normal values of E as obtained by tests in direct tension or compression, assigning values to F_c approximately as shown by the higher tests of short columns, and to F_t as given by direct tests of tensile strength, the writer has found that the lower limit of column strength is given fairly by the formulas when $\frac{c\,\varepsilon}{r^2} = 0.6$; and it also appears that the upper limit given by the formulas, when $\frac{c\,\varepsilon}{r^2} = 0.15$ is rarely exceeded.

These values are applicable alike to cast iron, wrought iron, mild steel, hard steel, and several kinds of timber, as will be seen on reference to the various diagrams of column tests appended.

It is interesting to note that in the case of solid, round columns, the value $\frac{c\varepsilon}{r^2} = 0.6$ for lower-limit strength corresponds to an equivalent eccentricity of (0.3 × radius of gyration), and applying this to a bar of 1 in. diameter, where $c = 0.5$ in., and $r = 0.25$ in., the value of ε would only be (0.3 × 0.25 in.) = 0.075 in.

Of this amount the possible errors in setting the test specimen may be only a small part, and if we assume that initial external curvature of bar, irregularities in the line of physical axis, and the effects of cold-straightening account for say 60% of the total value of ε, the remaining 40%, or 0.03 in., would be the extreme permissible amount of error in setting the specimen, and it becomes apparent how important such small errors are in experiments on columns, and how very carefully the testing must be carried out in order to develop even the lower-limit strength shown by the accompanying diagrams.

This also impresses the mind with the danger of generalizing from the results of any single series of tests where the number of tests is not very large.

Strictly speaking, the theory and resulting formulas of this paper apply only when the loads are such as will not stress the material of the column beyond the elastic limit, a condition which applies equally to the common theory of flexure of solid beams under simple bending. This, however, forms no bar to the extension of the application of the formula for elastic beam strength to cases of ultimate strength where the elastic limit is exceeded, provided proper recognition is given to the fact that the ultimate maximum fiber stresses apparently developed are not true values of tensile or compressive strength, but are always much higher than obtained from tests under direct stress. The reasons for this are now well known and need not be dealt with here.

There is, therefore, considerable justification for the application of the column formulas to a comparison with experiments on ultimate strength, which, in fact, form the only available basis of reference to which the engineer can appeal.

Differences of opinion exist among engineers as to whether columns should be designed with regard to the ultimate strength or with regard to the maximum fiber stress developed by the working load. The former is by far the more common practice, owing to the existing state of knowledge as to the principles of column strength, and this renders

it necessary for any theory and formulas, proposed for use in practice, to be compared with available experimental evidence.

In the writer's view, the more rational method is to design columns so as to ensure that given maximum fiber stresses will not be exceeded under the working load, while at the same time taking care to refer to experimental results, in order to see that a sufficient margin is provided against failure by instability in the longer lengths.

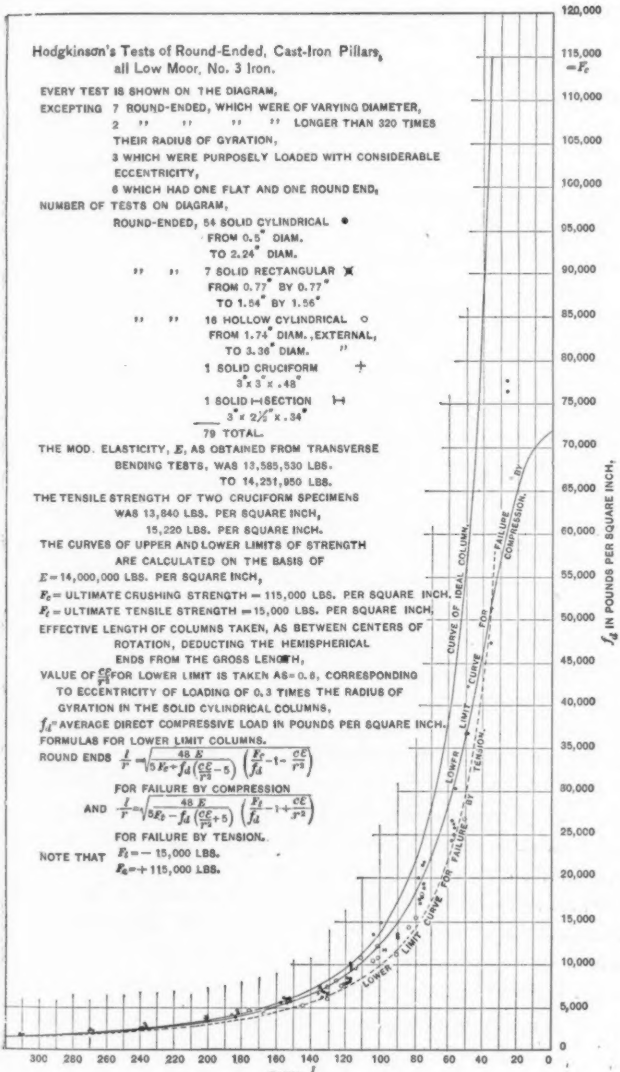
It would be reasonable to expect that the values of F_c and F_t, being maximum fiber stresses, would, when referred to tests carried to ultimate failure, partake somewhat of the nature and value generally accorded to the corresponding maximum apparent fiber stresses determined from tests of ultimate transverse strength of simple solid beams. It is necessary, however, to keep in view that it is hardly possible that they can have such high apparent values as in beam tests, owing to the rapidly accumulating bending moment developed in columns by the increasing deflection, which must evidently be intensified by any extra yielding of the extreme fibers due to their elastic limit being exceeded, and thus accelerating the deflection.

In the case of beams, of course, the increase in deflection accompanying an increasing load has no influence whatever on the bending moment.

The writer has not found it necessary for "centrally" loaded columns of wrought iron, mild or hard steel, and timber, to deal with the condition of failure by tensile stress, excepting as regards incipient tension in flat-ended columns.

The maximum compressive stress in a column of symmetrical section always has a greater value in pounds per square inch than the maximum tensile stress, and the difference between the compressive and tensile strengths must be considerable before tension becomes the controlling influence. This is the case with high-class cast iron, such as was used by Hodgkinson in his tests of Low Moor, No. 3 iron, and the curves for failure by tension are therefore plotted on the diagrams showing these tests.

In order to discover, if possible, any special features accompanying any particular set of tests, the writer has assigned a separate diagram to the principal sets of experiments where the number and range of tests were sufficiently great to justify it, and where this was not the case, the individual tests have been given distinguishing symbols to assist in arriving at a correct judgment.



Hodgkinson's Tests of Round-Ended, Cast-Iron Pillars,
all Low Moor, No. 3 Iron.
EVERY TEST IS SHOWN ON THE DIAGRAM,
EXCEPTING 7 ROUND-ENDED, WHICH WERE OF VARYING DIAMETER,
2 " " " " LONGER THAN 320 TIMES
THEIR RADIUS OF GYRATION,
3 WHICH WERE PURPOSELY LOADED WITH CONSIDERABLE
ECCENTRICITY,
6 WHICH HAD ONE FLAT AND ONE ROUND END,
NUMBER OF TESTS ON DIAGRAM,
ROUND-ENDED, 54 SOLID CYLINDRICAL
FROM 0.5" DIAM.
TO 2.24" DIAM.
" " 7 SOLID RECTANGULAR
FROM 0.77" BY 0.77"
TO 1.54" BY 1.56"
" " 16 HOLLOW CYLINDRICAL
FROM 1.74" DIAM., EXTERNAL,
TO 3.36" DIAM. "
1 SOLID CRUCIFORM
3" x 3" x .48"
1 SOLID ⊢⊣ SECTION
3" x 2½" x .34"
79 TOTAL.
THE MOD. ELASTICITY, E, AS OBTAINED FROM TRANSVERSE
BENDING TESTS, WAS 13,585,530 LBS.
TO 14,251,950 LBS.
THE TENSILE STRENGTH OF TWO CRUCIFORM SPECIMENS
WAS 13,840 LBS. PER SQUARE INCH,
15,220 LBS. PER SQUARE INCH.
THE CURVES OF UPPER AND LOWER LIMITS OF STRENGTH
ARE CALCULATED ON THE BASIS OF
E = 14,000,000 LBS. PER SQUARE INCH,
F_c = ULTIMATE CRUSHING STRENGTH = 115,000 LBS. PER SQUARE INCH,
F_t = ULTIMATE TENSILE STRENGTH = 15,000 LBS. PER SQUARE INCH,
EFFECTIVE LENGTH OF COLUMNS TAKEN, AS BETWEEN CENTERS OF
ROTATION, DEDUCTING THE HEMISPHERICAL
ENDS FROM THE GROSS LENGTH,
VALUE OF $\frac{c\mathcal{E}}{r^2}$ FOR LOWER LIMIT IS TAKEN AS = 0.6, CORRESPONDING
TO ECCENTRICITY OF LOADING OF 0.3 TIMES THE RADIUS OF
GYRATION IN THE SOLID CYLINDRICAL COLUMNS,
f_d = AVERAGE DIRECT COMPRESSIVE LOAD IN POUNDS PER SQUARE INCH.
FORMULAS FOR LOWER LIMIT COLUMNS.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48\,E}{5F_c + f_d\left(\frac{c\mathcal{E}}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\mathcal{E}}{r^2}\right)}$
FOR FAILURE BY COMPRESSION
AND $\frac{l}{r} = \sqrt{\frac{48\,E}{5F_t - f_d\left(\frac{c\mathcal{E}}{r^2} + 5\right)}\left(\frac{F_t}{f_d} - 1 + \frac{c\mathcal{E}}{r^2}\right)}$
FOR FAILURE BY TENSION.
NOTE THAT F_t = − 15,000 LBS.
F_c = + 115,000 LBS.
CURVE OF IDEAL COLUMN.
FAILURE BY COMPRESSION.
LOWER LIMIT CURVE FOR FAILURE BY TENSION.
LOWER LIMIT CURVE FOR FAILURE BY
120,000
115,000
$-F_c$
110,000
105,000
100,000
95,000
90,000
85,000
80,000
75,000
70,000
65,000
60,000
55,000
50,000
45,000
40,000
35,000
30,000
25,000
20,000
15,000
10,000
5,000
0
f_d IN POUNDS PER SQUARE INCH.
300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$

FIG. 17.

As has already been remarked, the writer has been unable to obtain practical verification of the precise influence of form of section from existing records of experiments. To demonstrate this influence by practical evidence it would be necessary to make a large number of new experiments on sections having widely different values of $\frac{c}{r}$, and at the same time to impose the loads with a comparatively large and predetermined value of eccentricity, in order to overshadow the relative influence of what has been dealt with in the foregoing pages as "equivalent eccentricity."

All the diagrams of column tests accompanying this paper have been made self-explanatory as far as possible.

They are as follows:

Cast Iron.

Fig. 17.—Representing 79 tests of round-ended columns of Low Moor, No. 3 cast iron, by Mr. Hodgkinson.

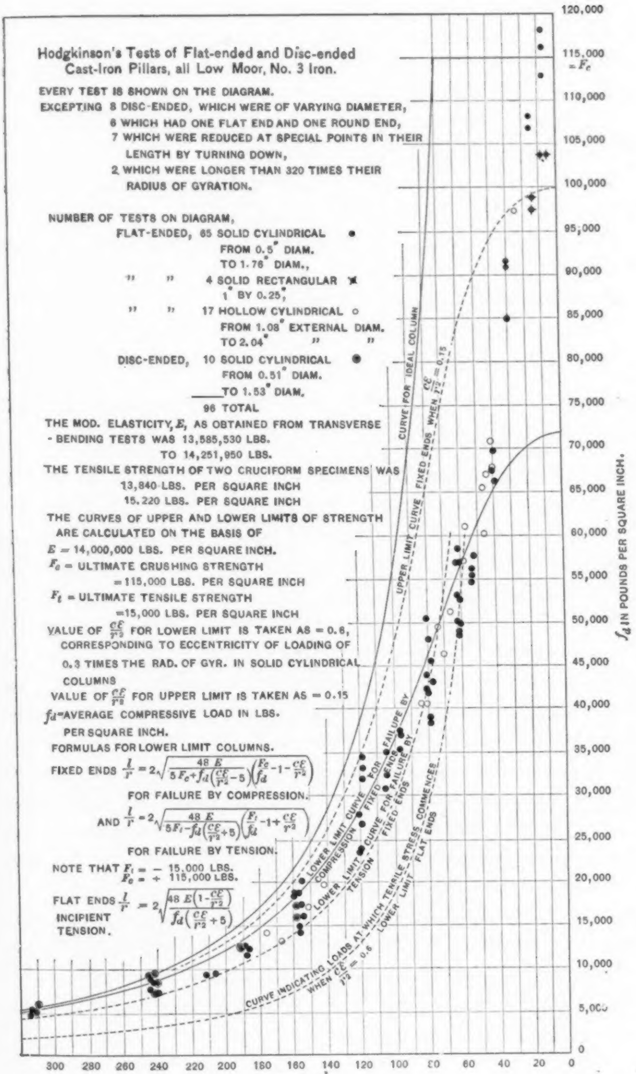
The tests are plotted to effective lengths, an allowance having been made for the rounding of the ends by taking them as hemispherical of the same diameter as the bars. The ends were not actually hemispherical in every case, some being somewhat more pointed, but no error of importance is involved in the assumption made.

Fig. 18. Representing 96 tests of flat-ended and disc-ended columns of Low Moor, No. 3 cast iron, by Mr. Hodgkinson.

It will be noticed in this set of tests on flat-ended columns, that the longer columns, from a ratio of 80 upward, do not show so low a strength as is indicated by the lowermost dotted curve for incipient tension, but failure at this critical point in flat-ended columns is evidenced so strongly in the case of tests of wrought iron and steel that the higher results obtained by Hodgkinson in these tests on cast iron must be attributed partly to his extreme care, and partly to the comparative fewness of tests of each length.

These two sets of tests have been abstracted from Mr. Hodgkinson's remarkably careful records.* They are the only published records of tests of cast-iron columns made in a consistent and scientific manner on one grade of material of known character. The values of load per square inch, radius of gyration and ratio $\frac{l}{r}$, have been calculated by the writer from Mr. Hodgkinson's figures.

* *Philosophical Transactions*, Royal Society of London, 1840.



Hodgkinson's Tests of Flat-ended and Disc-ended Cast-Iron Pillars, all Low Moor, No. 3 Iron.
EVERY TEST IS SHOWN ON THE DIAGRAM.
EXCEPTING 8 DISC-ENDED, WHICH WERE OF VARYING DIAMETER,
6 WHICH HAD ONE FLAT END AND ONE ROUND END,
7 WHICH WERE REDUCED AT SPECIAL POINTS IN THEIR LENGTH BY TURNING DOWN,
2 WHICH WERE LONGER THAN 320 TIMES THEIR RADIUS OF GYRATION.
NUMBER OF TESTS ON DIAGRAM,
FLAT-ENDED, 65 SOLID CYLINDRICAL FROM 0.5" DIAM. TO 1.76" DIAM.,
" " 4 SOLID RECTANGULAR 1" BY 0.25",
" " 17 HOLLOW CYLINDRICAL FROM 1.08" EXTERNAL DIAM. TO 2.04" " "
DISC-ENDED, 10 SOLID CYLINDRICAL FROM 0.51" DIAM. TO 1.53" DIAM.
96 TOTAL
THE MOD. ELASTICITY, E, AS OBTAINED FROM TRANSVERSE BENDING TESTS WAS 13,585,530 LBS. TO 14,251,950 LBS.
THE TENSILE STRENGTH OF TWO CRUCIFORM SPECIMENS WAS 13,840 LBS. PER SQUARE INCH 15,220 LBS. PER SQUARE INCH
THE CURVES OF UPPER AND LOWER LIMITS OF STRENGTH ARE CALCULATED ON THE BASIS OF
E = 14,000,000 LBS. PER SQUARE INCH.
F_c = ULTIMATE CRUSHING STRENGTH =115,000 LBS. PER SQUARE INCH
F_t = ULTIMATE TENSILE STRENGTH =15,000 LBS. PER SQUARE INCH
VALUE OF $\frac{c\mathcal{E}}{r^2}$ FOR LOWER LIMIT IS TAKEN AS = 0.6, CORRESPONDING TO ECCENTRICITY OF LOADING OF 0.3 TIMES THE RAD. OF GYR. IN SOLID CYLINDRICAL COLUMNS
VALUE OF $\frac{c\mathcal{E}}{r^2}$ FOR UPPER LIMIT IS TAKEN AS = 0.15
f_d=AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
FORMULAS FOR LOWER LIMIT COLUMNS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48\,E}{5F_c + f_d\left(\frac{c\mathcal{E}}{r^2} - 5\right)\left(\frac{F_c}{f_d} - 1 - \frac{c\mathcal{E}}{r^2}\right)}}$
FOR FAILURE BY COMPRESSION.
AND $\frac{l}{r} = 2\sqrt{\frac{48\,E}{5F_t - f_d\left(\frac{c\mathcal{E}}{r^2} + 5\right)\left(\frac{F_t}{f_d} - 1 + \frac{c\mathcal{E}}{r^2}\right)}}$
FOR FAILURE BY TENSION.
NOTE THAT F_t = − 15,000 LBS. F_c = + 115,000 LBS.
FLAT ENDS INCIPIENT TENSION. $\frac{l}{r} = 2\sqrt{\frac{48\,E\left(1 - \frac{c\mathcal{E}}{r^2}\right)}{f_d\left(\frac{c\mathcal{E}}{r^2} + 5\right)}}$
CURVE FOR IDEAL COLUMN
UPPER LIMIT CURVE FIXED ENDS WHEN $\frac{c\mathcal{E}}{r^2} = 0.15$
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION FIXED ENDS
LOWER LIMIT CURVE FOR FAILURE BY TENSION FIXED ENDS
LOWER LIMIT FLAT ENDS
CURVE INDICATING LOADS AT WHICH TENSILE STRESS COMMENCES WHEN $\frac{c\mathcal{E}}{r^2} = 0.6$
120,000
115,000 $=F_c$
110,000
105,000
100,000
95,000
90,000
85,000
80,000
75,000
70,000
65,000
60,000
55,000
50,000
45,000
40,000
35,000
30,000
25,000
20,000
15,000
10,000
5,000
0
f_d IN POUNDS PER SQUARE INCH.
300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$

FIG. 18.

Fig. 19.—Representing 76 tests of cylindrical, flat-ended columns of cast iron of various kinds, by Mr. Hodgkinson.*

One test by Mr. Charnock, at Bradford Technical College, England, on a hollow cast iron column, with flat flanged ends.†

One test by Professor John Goodman, at Yorkshire College, Leeds, England, on a hollow column with flanged ends.‡

Fourteen tests on cylindrical, hollow columns, with flat ends, made by the New York City Department of Buildings.§

Fourteen tests on hollow columns with flat and flanged ends, at Watertown Arsenal.||

Total number of tests on diagram, 106.

The sectional areas, loads per square inch, radii of gyration and values of $\frac{l}{r}$, have all been calculated by the writer from the figures given in the records, except for the New York tests, for which the radii of gyration and ratio $\frac{l}{r}$, only, were calculated.

Most of the Watertown tests were on tapered columns, and all the areas, loads per square inch and radii of gyration calculated for these refer to the section at the middle of the column length, which will explain the divergence from the figures for ultimate strength per square inch given by Professor Lanza.

The results plotted on Fig. 19 refer to tests of columns of various kinds of cast iron. Hodgkinson's tests alone cover 17 different irons, of widely different compressive strength, in short specimens. No information is given as to the physical characteristics of the irons used in the other tests plotted on the diagram.

A number of the columns of this (1857) series by Hodgkinson were subjected to more than one test. They were made long at first, and after being tested they were cut down into shorter columns and re-tested, a circumstance still further adding to the difficulty of deriving any definite laws of strength from these tests.

Regarded as a means of determining the influence of column proportions on ultimate strength, the results of any of the tests plotted on Fig. 19 are of little value, and a most cursory consideration will

*Abstracted from Mr. Hodgkinson's paper in *Philosophical Transactions*, Royal Society of London, 1857.

† *Engineering*, February 28th, 1896.

‡ *Engineering*, September 11th, 1896.

§ *Engineering News*, January 13th, 1898.

|| Reports, 1887-1888, and Lanza's "Applied Mechanics."

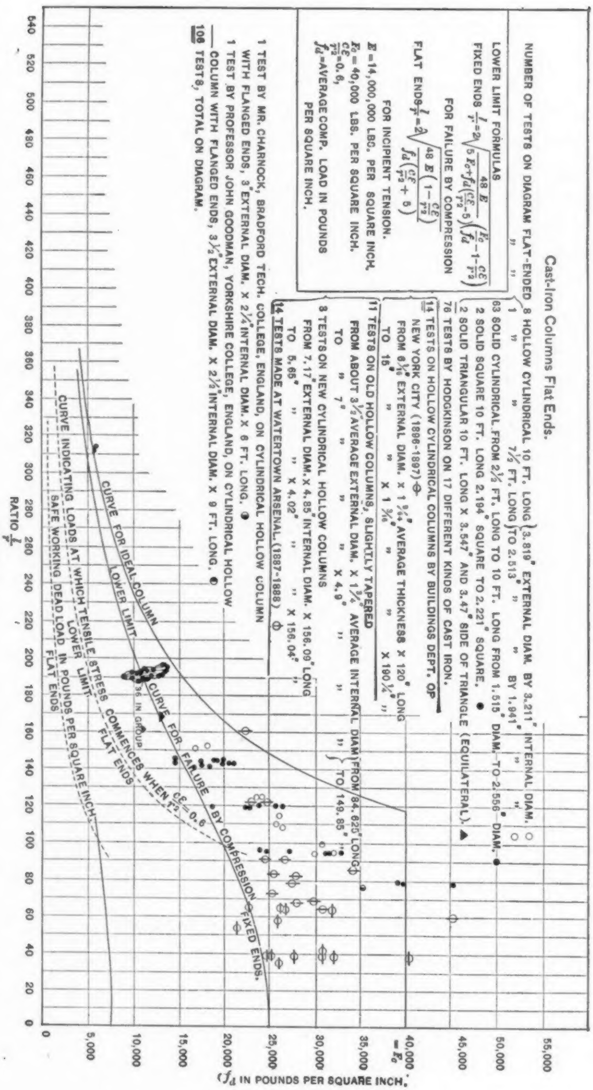

Cast-Iron Columns Flat Ends.
NUMBER OF TESTS ON DIAGRAM FLAT-ENDED 8 HOLLOW CYLINDRICAL 10 FT. LONG 3.819" EXTERNAL DIAM. BY 3.211" INTERNAL DIAM.
1 " " 7½ FT. LONG TO 2.513" " " BY 1.941" " "
63 SOLID CYLINDRICAL FROM 2½ FT. LONG TO 10 FT. LONG FROM 1.515" DIAM. TO 2.556" DIAM.
2 SOLID SQUARE 10 FT. LONG 2.194" SQUARE TO 2.221" SQUARE.
2 SOLID TRIANGULAR 10 FT. LONG X 3.547" AND 3.47" SIDE OF TRIANGLE (EQUILATERAL).
76 TESTS BY HODGKINSON ON 17 DIFFERENT KINDS OF CAST IRON.
14 TESTS ON HOLLOW CYLINDRICAL COLUMNS BY BUILDINGS DEPT. OF NEW YORK CITY (1896-1897)
FROM 6 5/16" EXTERNAL DIAM. X 1 5/64" AVERAGE THICKNESS X 120" LONG
TO 15" " " X 1 3/16" " " X 190¼" "
11 TESTS ON OLD HOLLOW COLUMNS, SLIGHTLY TAPERED
FROM ABOUT 3½" AVERAGE EXTERNAL DIAM. X 1¾" AVERAGE INTERNAL DIAM. FROM 84.625" LONG
TO " 7" " " " X 4.9" " " " TO 149.85" "
8 TESTS ON NEW CYLINDRICAL HOLLOW COLUMNS
FROM 7.17" EXTERNAL DIAM. X 4.85" INTERNAL DIAM. X 156.09" LONG
TO 5.65" " " X 4.02" " " X 156.04" "
14 TESTS MADE AT WATERTOWN ARSENAL. (1887-1888)
1 TEST BY MR. CHARNOCK, BRADFORD TECH. COLLEGE, ENGLAND, ON CYLINDRICAL HOLLOW COLUMN WITH FLANGED ENDS, 3" EXTERNAL DIAM. X 2¼" INTERNAL DIAM. X 6 FT. LONG.
1 TEST BY PROFESSOR JOHN GOODMAN, YORKSHIRE COLLEGE, ENGLAND, ON CYLINDRICAL HOLLOW COLUMN WITH FLANGED ENDS, 3½" EXTERNAL DIAM. X 2½" INTERNAL DIAM. X 9 FT. LONG.
106 TESTS, TOTAL ON DIAGRAM.
LOWER LIMIT FORMULAS
FIXED ENDS $\frac{l}{r}=2\sqrt{\frac{48\ E}{5\ F_c+f_d\left(\frac{c\varepsilon}{r^2}-5\right)}\left(\frac{F_c}{f_d}-1-\frac{c\varepsilon}{r^2}\right)}$
FOR FAILURE BY COMPRESSION
FLAT ENDS $\frac{l}{r}=2\sqrt{\frac{48\ E\left(1-\frac{c\varepsilon}{r^2}\right)}{f_d\left(\frac{c\varepsilon}{r^2}+5\right)}}$
FOR INCIPIENT TENSION.
E = 14,000,000 LBS. PER SQUARE INCH.
F_c = 40,000 LBS. PER SQUARE INCH.
$\frac{c\varepsilon}{r^2}$ = 0.6,
f_d = AVERAGE COMP. LOAD IN POUNDS PER SQUARE INCH.
CURVE FOR IDEAL COLUMN
LOWER LIMIT
CURVE FOR FAILURE BY COMPRESSION
FIXED ENDS.
36 IN GROUP
CURVE INDICATING LOADS AT WHICH TENSILE STRESS COMMENCES WHEN $\frac{c\varepsilon}{r^2}$ = 0.6
LOWER LIMIT
FLAT ENDS
SAFE WORKING DEAD LOAD IN POUNDS PER SQUARE INCH
FLAT ENDS
55,000
50,000
45,000
40,000 = F_c
35,000
30,000
25,000
20,000
15,000
10,000
5,000
0
f_d IN POUNDS PER SQUARE INCH.
540 520 500 480 460 440 420 400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$

FIG. 19.

show the absurdity of attempting to generalize with regard to any of the three principal sets of tests of this diagram. Nevertheless, these are the most important tests of cast-iron columns yet made, from the point of view of the engineer, representing, as they do, all the experimental evidence at present available to justify the confidence of the designer using this material in its commoner qualities without any definite knowledge or check upon its physical characteristics.

It is worthy of note that prior to the Watertown tests of 1887 and 1888, there was absolutely no published experimental evidence existing of the strength of common grade cast-iron columns of the proportions of length to radius of gyration in most common use. Hodgkinson's tests (1857) did not give results on columns of shorter lengths than 79 or 80 times the radius of gyration, and his tests in the 1840 paper were on iron of a comparatively high class. This want has, to a slight extent, been filled by the Watertown tests and the New York Building Department tests.

It is surprising to think of the enormous number of cast-iron columns which have been put into use without any justification for the loads imposed on them, except a simple faith in Hodgkinson's, Gordon's and Rankine's formulas, and in the numerous tables calculated therefrom and published in engineering pocketbooks and treatises.

The writer does not pretend that the curves plotted on Fig. 19 have any other than a purely accidental correspondence with the experimental results shown on the diagram.

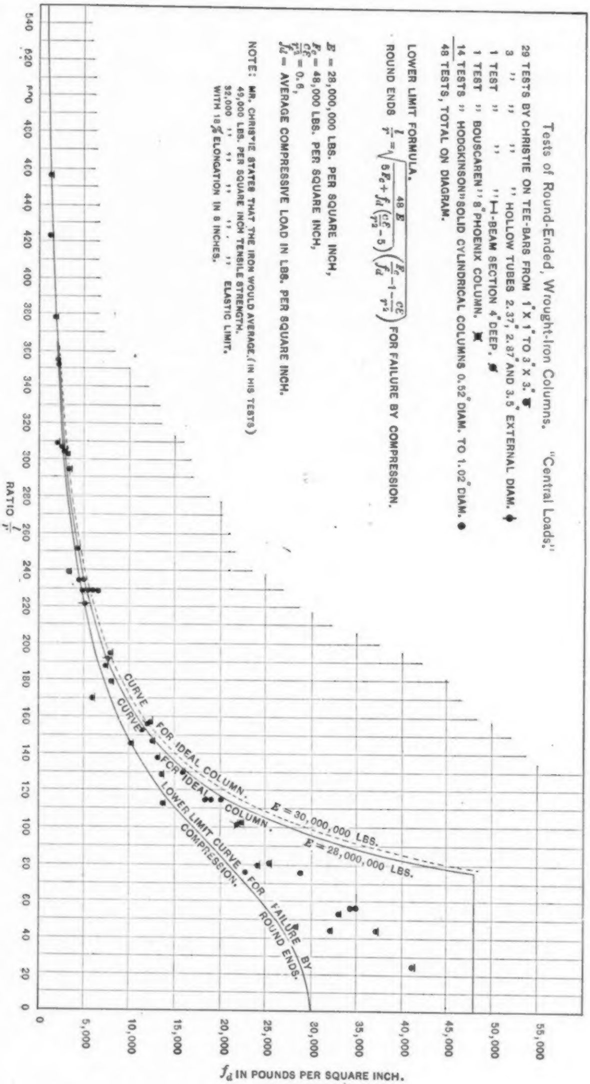
The curves calculated from the formula refer to material of certain fixed characteristics, while the experiments plotted on Fig. 19 were made on cast iron of widely different grades.

At the same time the curves follow definite laws, and may serve as a basis of reference. The writer, in his own practice, would not care to count upon higher ultimate strengths for common cast-iron columns than are given by the lowest curves on the diagram.

Wrought Iron.—Round or Pivoted Ends.

Fig. 20.—Representing 33 tests of round-ended columns by Mr. Christie.* These tests have all been plotted to lengths measured from center to center of hemispherical ends. Mr. Christie gives values of R based on extreme lengths.

* *Transactions*, Am. Soc. C. E., Vol. xiii.



Tests of Round-Ended, Wrought-Iron Columns. "Central Loads."
29 TESTS BY CHRISTIE ON TEE-BARS FROM 1" X 1" TO 3" X 3".
3 " " " HOLLOW TUBES 2.37", 2.87" AND 3.5" EXTERNAL DIAM.
1 TEST " " "I-BEAM SECTION 4" DEEP.
1 TEST " BOUSCAREN'S 8" PHOENIX COLUMN.
14 TESTS " HODGKINSON'S SOLID CYLINDRICAL COLUMNS 0.52" DIAM. TO 1.02" DIAM.
48 TESTS, TOTAL ON DIAGRAM.
LOWER LIMIT FORMULA.
ROUND ENDS $\frac{l}{r}=\sqrt{\frac{48E}{\left(5F_c+f_d\left(\frac{cc}{r^2}-5\right)\right)\left(\frac{F_c}{f_d}-1-\frac{cc}{r^2}\right)}}$ FOR FAILURE BY COMPRESSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = 48,000 LBS. PER SQUARE INCH,
$\frac{cc}{r^2}$ = 0.6,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
NOTE: MR. CHRISTIE STATES THAT THE IRON WOULD AVERAGE, (IN HIS TESTS)
48,000 LBS. PER SQUARE INCH TENSILE STRENGTH.
32,000 " " " " ELASTIC LIMIT.
WITH 18% ELONGATION IN 8 INCHES.
CURVE FOR IDEAL COLUMN.
E = 30,000,000 LBS.
CURVE FOR IDEAL COLUMN.
E = 28,000,000 LBS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. ROUND ENDS.
RATIO $\frac{l}{r}$
f_d IN POUNDS PER SQUARE INCH.

Fig. 20.

In the case of Tests Nos. 227, 228 and 229 the ratio $\frac{l}{r}$ has been recalculated from the lengths given in Mr. Christie's Table No. 6. The values of $\frac{l}{r}$ for these three tests given in this table do not agree with the lengths and radii of gyration.

The 33 tests by Mr. Christie shown on this diagram are Nos. 200 to 229, inclusive, of his Table No. 6, and Nos. 287, 290 and 293 of his Table No. 8.

One test of a round-ended column of large size by L. F. G. Bouscaren,* M. Am. Soc. C. E. The ends of this column were portions of a sphere of 10⅛ ins. radius, and the effective length has, therefore, been taken to be 20¼ ins. shorter than the length over all.

Fourteen tests of round-ended columns by Mr. Hodgkinson.† Allowance for the round ends, in arriving at effective length, has been made in this case also. The sectional areas, loads per square inch, radii of gyration, and values of $\frac{l}{r}$, have been calculated by the writer from Mr. Hodgkinson's records.

Total number of tests on diagram, 48.

Fig. 21.—Representing 116 results, by Professor Tetmajer, on pivot-ended columns.‡ These results represent 210 experiments. Ninety-four of the results plotted represent in each case the average of two tests, while twenty-two of the results are for single experiments. The results, as plotted by the writer, are in every case for the load per gross square inch. Some of the specimens were compounded of two or four pieces riveted together, and in the diagram given by Professor J. B. Johnson in his "Materials of Construction," these results appear to have been plotted for load per net square inch (the rivet holes being deducted), and they have, therefore, too high a value. This error was repeated in the reproduction of the diagram by Mr. Marston in his discussion of Professor Cain's paper.§

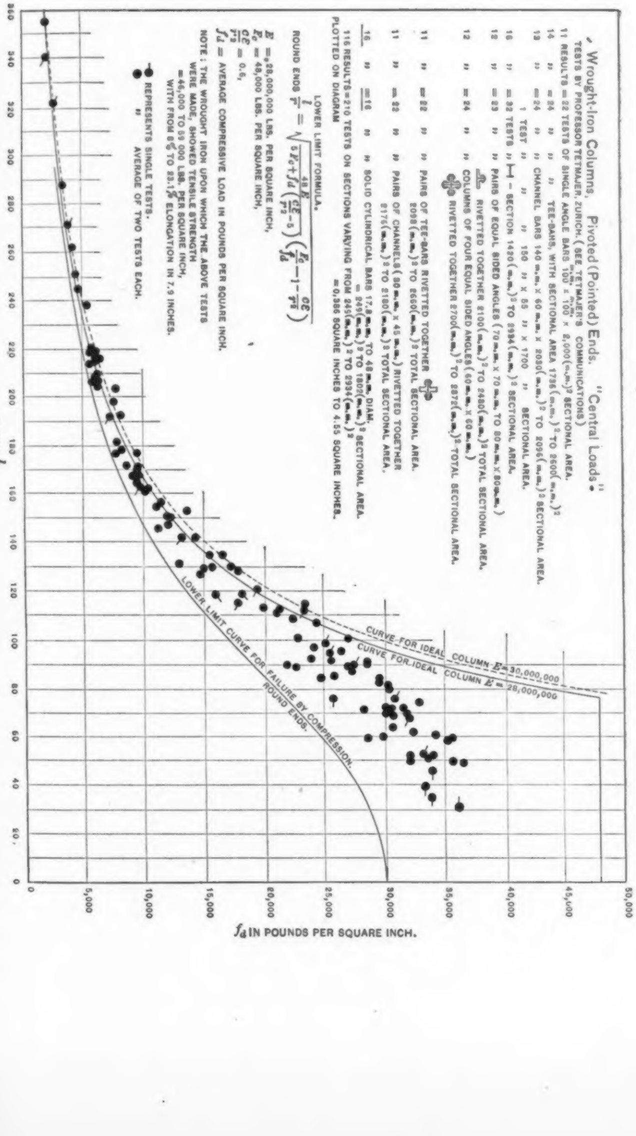
Fig. 22.—This diagram is simply a combination of Figs. 20 and 21, and therefore includes practically the whole of the available experimental evidence as to the strength of wrought-iron columns with both ends free, and unconstrained, *i. e.*, with "round" or "pivot ends" and with "central" loading.

* *Transactions*, Am. Soc. C. E., Vol. ix.

† *Philosophical Transactions*, Royal Society, London, 1840.

‡ "Tetmajer's Communications."

§ "The Ideal Column," *Transactions*, Am. Soc. C. E., Vol. xxxix, pp. 109 to 111.



RATIO $\frac{l}{r}$

FIG. 21.

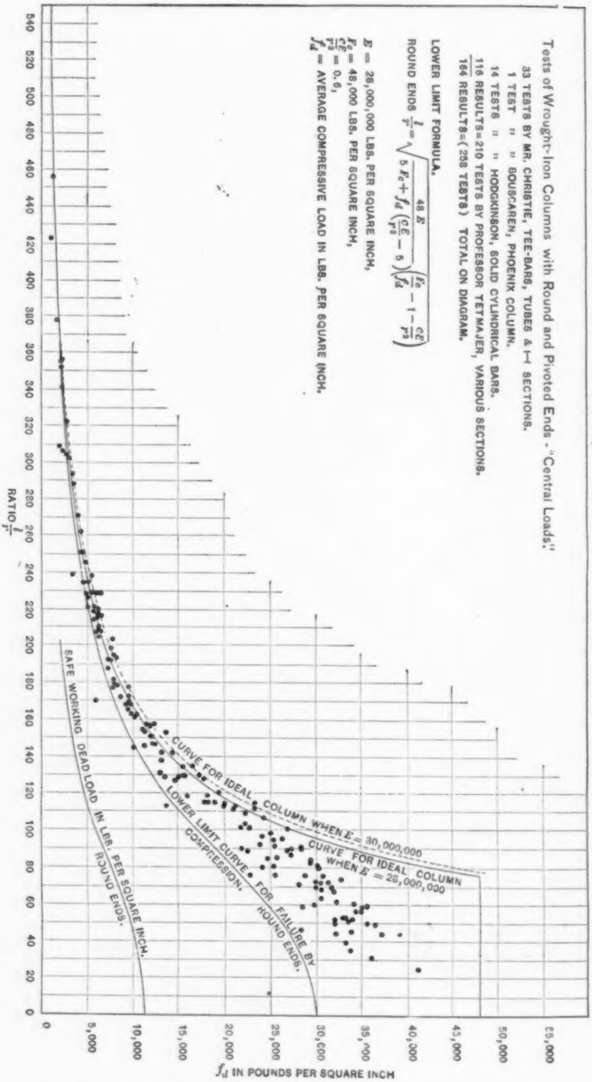

Tests of Wrought-Iron Columns with Round and Pivoted Ends - "Central Loads."
33 TESTS BY MR. CHRISTIE, TEE-BARS, TUBES & I-I SECTIONS.
1 TEST " " BOUSCAREN, PHOENIX COLUMN.
14 TESTS " " HODGKINSON, SOLID CYLINDRICAL BARS.
116 RESULTS=210 TESTS BY PROFESSOR TETMAJER, VARIOUS SECTIONS.
164 RESULTS=(258 TESTS) TOTAL ON DIAGRAM.
LOWER LIMIT FORMULA.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48\,E}{5\,F_c + f_d\left(\frac{cE}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{cE}{r^2}\right)}$
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = 48,000 LBS. PER SQUARE INCH,
$\frac{cE}{r^2}$ = 0.6,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN WHEN E = 30,000,000
CURVE FOR IDEAL COLUMN WHEN E = 28,000,000
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. ROUND ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. ROUND ENDS.
RATIO $\frac{l}{r}$
f_d IN POUNDS PER SQUARE INCH

FIG. 22.

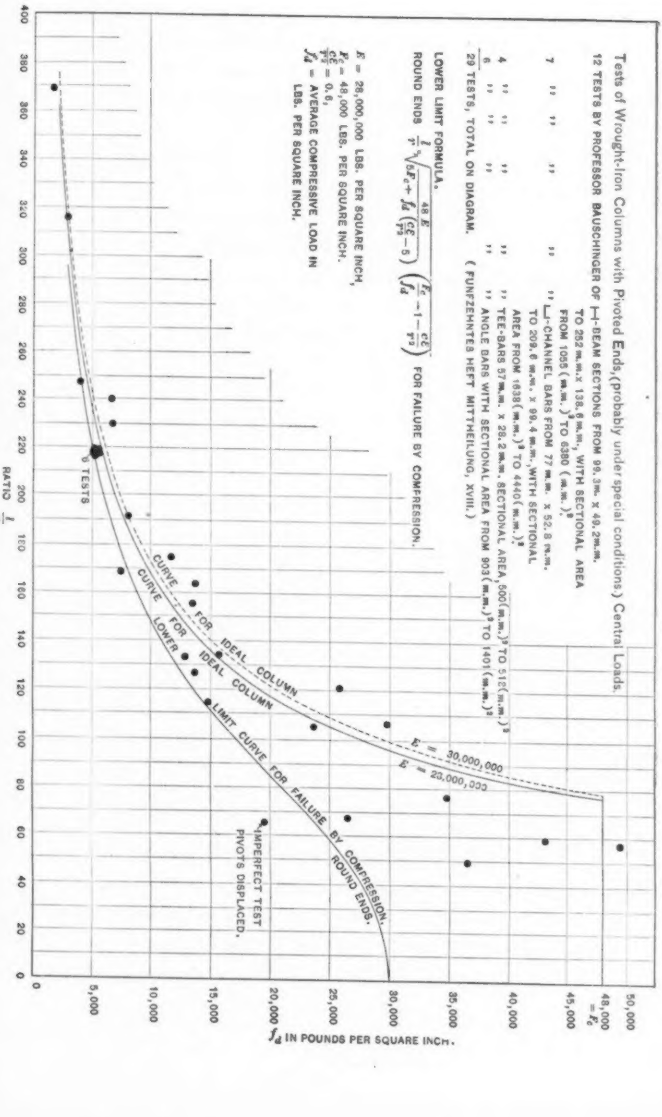

Tests of Wrought-Iron Columns with Pivoted Ends, (probably under special conditions.) Central Loads.
12 TESTS BY PROFESSOR BAUSCHINGER OF I-BEAM SECTIONS FROM 99.3 m.m. x 49.2 m.m. TO 252 m.m. x 138.6 m.m., WITH SECTIONAL AREA FROM 1055 (m.m.)² TO 6380 (m.m.)².
7 " " " " ⊔-CHANNEL BARS FROM 77 m.m. x 52.8 m.m. TO 209.6 m.m. x 99.4 m.m., WITH SECTIONAL AREA FROM 1638 (m.m.)² TO 4440 (m.m.)².
4 " " " " TEE-BARS 57 m.m. x 28.2 m.m. SECTIONAL AREA, 500 (m.m.)² TO 512 (m.m.)².
6 " " " " ANGLE BARS WITH SECTIONAL AREA FROM 903 (m.m.)² TO 1401 (m.m.)².
29 TESTS, TOTAL ON DIAGRAM. (FUNFZEHNTES HEFT MITTHEILUNG, XVIII.)
LOWER LIMIT FORMULA.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48\,E}{5F_c + f_d\left(\frac{c\epsilon}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\epsilon}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = 48,000 LBS. PER SQUARE INCH.
$\frac{c\epsilon}{r^2}$ = 0.6,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN
CURVE FOR IDEAL COLUMN
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. ROUND ENDS.
E = 30,000,000
E = 20,000,000
6 TESTS
IMPERFECT TEST PIVOTS DISPLACED.
RATIO $\frac{l}{r}$
400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 48,000 $=F_c$ 50,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 28.

The experiments of M. Considère on pivot-ended columns cannot be made use of as bases of reference for practical work, on account of the attempt made to feel for the "physical axis," or axis of greatest resistance, by moving the pivot ends out of the geometric axis, a refinement which is entirely out of the question in practical construction.

This objection probably applies also to Professor Bauschinger's experiments on pivot-ended columns, but this point the writer has not been able to make out clearly from the records.

It may also be noted that M. Considère's test specimens were of very small sections, the heaviest being only 11 mm. × 23 mm. (rectangular) or 253 sq. mm. = about 0.4 sq. in., and the lightest only 77 sq. mm. (angle section) = about 0.12 sq. in.* This last objection does not apply to Bauschinger's pivot-ended tests, in which the sections ranged from a maximum of 63.8 sq. cm. (⊢⊣-section, 25.2 cm. × 13.86 cm.), or about 9.9 sq. ins. sectional area, to a minimum of 5 sq. cm. (T-bar, 5.7 cm. × 2.82 cm.), or about 0.755 sq. in. sectional area.

Although it has been stated that Bauschinger's tests (pivot-ended) probably cannot be used as bases of reference in practical work, owing to the refinements probably adopted in carrying out the tests, yet it has seemed to the writer that a record of them is necessary to the completeness of this paper; and a still stronger reason for their presentation in diagram form lies in the fact that in spite of all the highly skilled care and accuracy bestowed on the tests, Bauschinger did not succeed in keeping the lower tests above the lower limit found in the tests of other experimenters, as will be seen on reference to Fig. 23.

Fig. 23.—Twenty-nine tests by Professor Bauschinger on pivot-ended columns of wrought iron.†

Fig. 24.—Nine results, representing eighteen experiments by Professor Tetmajer on pivot-ended wrought-iron columns under intentionally eccentric loads. These results are in each case the average of the tests of two specimens.‡ It is unfortunate that the record of each individual experiment has not been given by Professor Tetmajer.

* Considère's Report on "La Resistance au Flambement des Pièces Comprimées" (French Commission des Methodes D'Essai des Materiaux de Construction. Tome iii).

† Fünfzehntes Heft, *Mittheilung*, xviii.

‡ "Tetmajer's Communications."

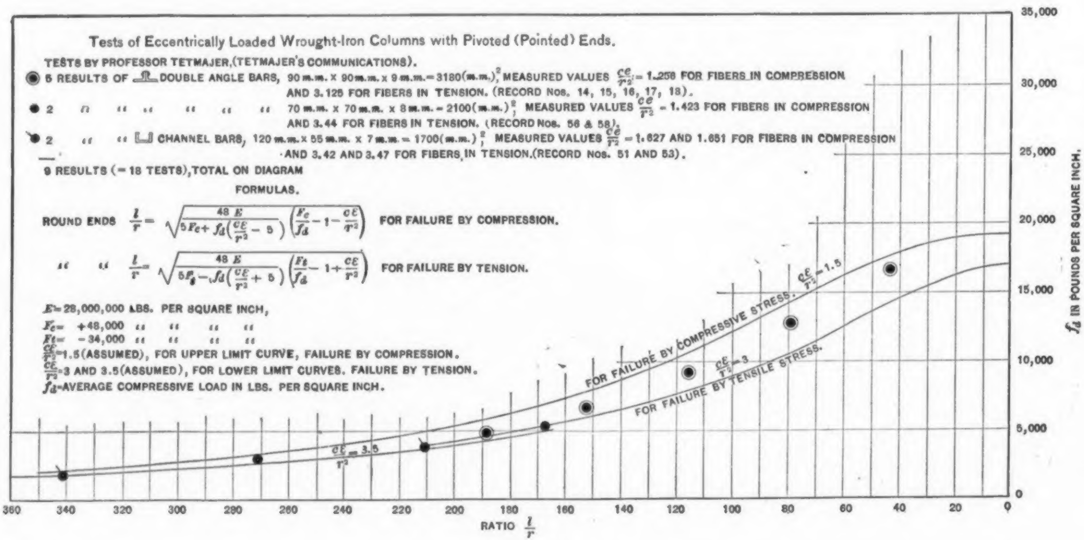

Tests of Eccentrically Loaded Wrought-Iron Columns with Pivoted (Pointed) Ends.
TESTS BY PROFESSOR TETMAJER,(TETMAJER'S COMMUNICATIONS).
5 RESULTS OF DOUBLE ANGLE BARS, 90 m.m. x 90 m.m. x 9 m.m. = 3180 (m.m.)^2; MEASURED VALUES $\frac{ce}{r^2}$ = 1.258 FOR FIBERS IN COMPRESSION AND 3.125 FOR FIBERS IN TENSION. (RECORD Nos. 14, 15, 16, 17, 18).
2 " " " " " 70 m.m. x 70 m.m. x 8 m.m. = 2100 (m.m.)^2; MEASURED VALUES $\frac{ce}{r^2}$ = 1.423 FOR FIBERS IN COMPRESSION AND 3.44 FOR FIBERS IN TENSION. (RECORD Nos. 56 & 58).
2 " " CHANNEL BARS, 120 m.m. x 55 m.m. x 7 m.m. = 1700 (m.m.)^2; MEASURED VALUES $\frac{ce}{r^2}$ = 1.627 AND 1.651 FOR FIBERS IN COMPRESSION AND 3.42 AND 3.47 FOR FIBERS IN TENSION. (RECORD Nos. 51 AND 53).
9 RESULTS (= 18 TESTS), TOTAL ON DIAGRAM
FORMULAS.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48\,E}{5F_c + f_a\left(\frac{ce}{r^2} - 5\right)}\left(\frac{F_c}{f_a} - 1 - \frac{ce}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
" " $\frac{l}{r} = \sqrt{\frac{48\,E}{5F_t - f_a\left(\frac{ce}{r^2} + 5\right)}\left(\frac{F_t}{f_a} - 1 + \frac{ce}{r^2}\right)}$ FOR FAILURE BY TENSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = +48,000 " " " "
F_t = −34,000 " " " "
$\frac{ce}{r^2}$ = 1.5 (ASSUMED), FOR UPPER LIMIT CURVE, FAILURE BY COMPRESSION.
$\frac{ce}{r^2}$ = 3 AND 3.5 (ASSUMED), FOR LOWER LIMIT CURVES. FAILURE BY TENSION.
f_a = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
FOR FAILURE BY COMPRESSIVE STRESS. $\frac{ce}{r^2}$ = 1.5
FOR FAILURE BY TENSILE STRESS.
$\frac{ce}{r^2}$ = 3
$\frac{ce}{r^2}$ = 3.5
360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000
f_a IN POUNDS PER SQUARE INCH.

FIG. 24

The record numbers of the results are noted on the diagram. Tetmajer's tests (non-axial), Nos. 27 to 34 inclusive, have not been plotted by the writer, as they were on T-bars, 100 mm. × 100 mm. × 10 mm., with the load imposed eccentrically in the line of the greatest radius of gyration, while the bars all failed in the direction of the least radius of gyration. It may be remarked that the amount of the intentional eccentricity in these tests (Nos. 27 to 34) was not sufficiently great in any case to ensure that failure would occur by flexure in the plane of the greatest radius of gyration, and the "accidental" equivalent eccentricity in the plane of the least radius of gyration was evidently the controlling factor.

The examples on the diagram have been selected for the sole reason that they were most nearly uniform in the character of section, method of loading, and amount of eccentricity.

Fourteen of the tests were on pairs of angle bars riveted together to form a T-section, and the remaining four were on channel bars.

In each case the eccentricity of loading was such that the tables of the T-sections or channels were subjected to the greatest compressive stress, and in consequence, the value of $\frac{c\,e}{r^2}$ was much less when referred to the table faces than when referred to the points of the legs of the T-sections or channels, rendering it necessary to use two different values of $\frac{c\,\varepsilon}{r^2}$, when plotting the curves by the writer's formula, one being for failure by compressive stress in the table faces, and the other for failure by tensile stress in the points of the legs.

The values of $\frac{c\,e}{r^2}$ deduced from the sections and the actual value of intentional eccentricity were as follows:

For results.	$\frac{c\,e}{r^2}$ for Compressive stress.	$\frac{c\,e}{r^2}$ for Tensile stress
Nos. 14, 15, 16, 17 and 18	1.256	3.125
Nos. 56 and 58	1.423	3.440
Nos. 51 and 52	1.627 and 1.651	3.420 and 3.470

The curves plotted on the diagram have been calculated from values:

	For Compressive stress.	For Tensile stress.
$\frac{c\,\varepsilon}{r^2}$	1.5	3 and 3.5

and the characteristic agreement is sufficiently satisfactory when it is

remembered that we are dealing with tests of ultimate strength, and that each result plotted on the diagram represents the average of two tests.

The other non-axial tests made by Tetmajer had values $\frac{c\,e}{r^2}$ of too little uniformity to allow of them being used to illustrate the influence

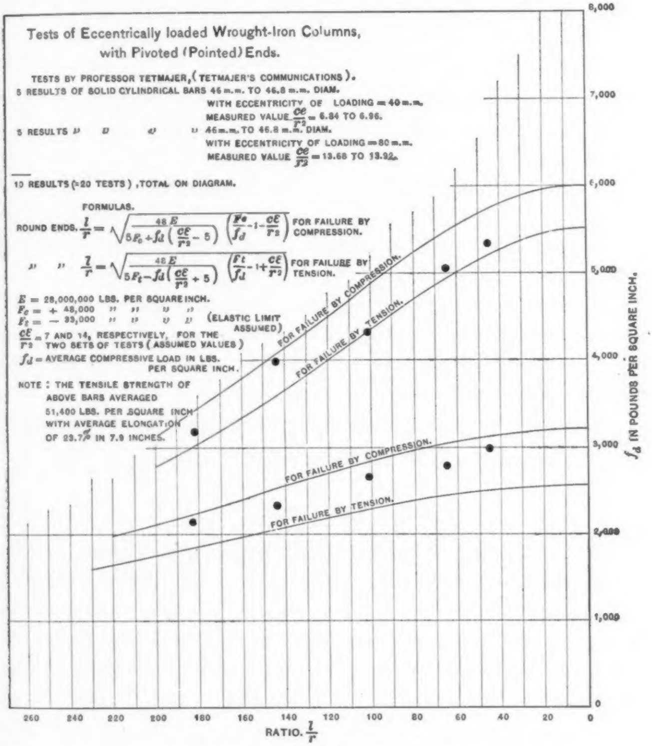


FIG. 25.

of eccentric loading combined with the influence of varying ratios of $\frac{l}{r}$. They do, however, most clearly show the serious loss of strength caused by eccentric loading.

Fig. 25.—Ten results, representing twenty tests by Professor Tetmajer on pivot-ended wrought-iron columns under intentionally

eccentric loads. Each result plotted is the average of two tests.* These results are especially interesting, notwithstanding their small number, as the tests were on solid, round bars of one make of iron throughout, and the observed and intentional value of eccentricity of loading was very large, and thus greatly overshadowed accidental conditions.

Ten of the experiments were made with a value of $\frac{c\,e}{r^2} = 6.84$ to 6.96, and the other ten with a value of $\frac{c\,e}{r^2} = 13.68$ to 13.92. The ultimate strength of the iron under direct tension is given by Tetmajer as 51 400 lbs. per square inch, with an ultimate elongation of 23.7% in 200 mm. ($7\frac{7}{8}$ ins. nearly).

In this diagram the vertical scale of the load has been made much larger than in the other diagrams in order to emphasize the difference in the results, and to show their characteristic agreement with the writer's calculated curves.

In each of the two sets of experiments plotted on this diagram the upper curve indicates the loads causing a maximum compressive stress of 48 000 lbs. per square inch, and the lower curve indicates the loads causing a maximum tensile fiber stress of 33 000 lbs. per square inch.

The curves for the upper set of tests have been plotted for a value of $\frac{c\,\varepsilon}{r^2} = 7$, and those for the lower set for a value of 14.

Wrought Iron.—Fixed Ends.

Fig. 26.—Twenty-five tests by Mr. Christie on fixed-ended columns of angle bars.† Reference has already been made to the difficulty of realizing fixity of ends in columns, and Mr. Christie remarks in his description of his experiments, that the lengths of the fixed-ended struts were measured between the clamps, whereas the point of absolute fixing probably occurred at some place within the clamps, and the values given for the ratio $\frac{l}{r}$ would then be too low.

If Mr. Christie's suggestion were adopted and a somewhat higher value assumed for the value of $\frac{l}{r}$, the already fairly satisfactory agreement between the writer's lower-limit curve and the lower results of the experiments would be still more pronounced.

* "Tetmajer's Communications."

† *Transactions*, Am. Soc. C. E., Vol. xiii.

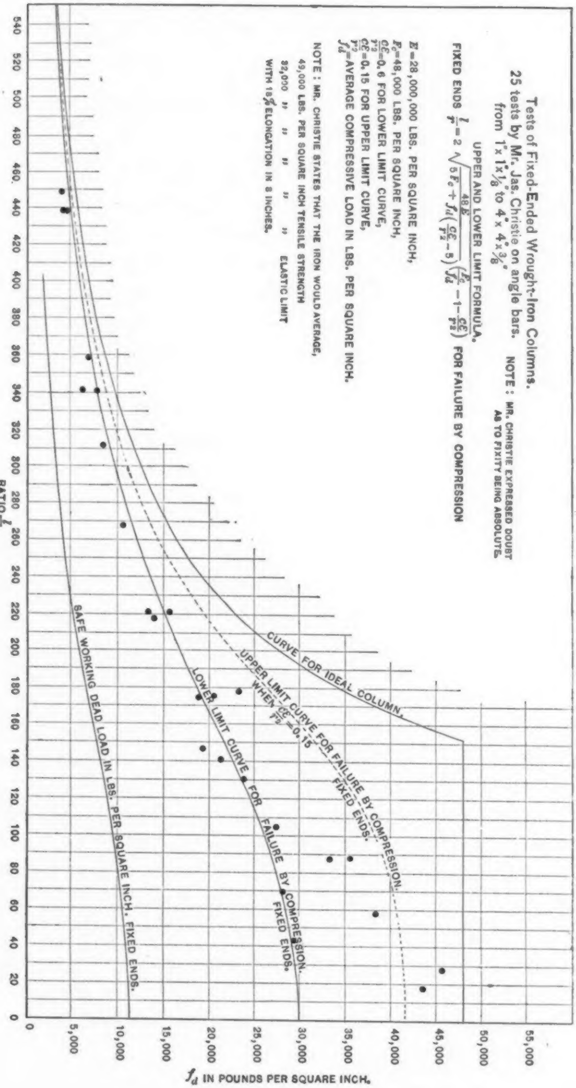

Tests of Fixed-Ended Wrought-Iron Columns.
25 tests by Mr. Jas. Christie on angle bars.
from 1"x 1"x 1/8" to 4" x 4" x 3/8"
NOTE: MR. CHRISTIE EXPRESSED DOUBT AS TO FIXITY BEING ABSOLUTE.
UPPER AND LOWER LIMIT FORMULA.
FOR FAILURE BY COMPRESSION
E=28,000,000 LBS. PER SQUARE INCH.
Fc=48,000 LBS. PER SQUARE INCH.
=0.6 FOR LOWER LIMIT CURVE,
=0.15 FOR UPPER LIMIT CURVE,
fd=AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
NOTE: MR. CHRISTIE STATES THAT THE IRON WOULD AVERAGE,
49,000 LBS. PER SQUARE INCH TENSILE STRENGTH
32,000 " " " " ELASTIC LIMIT
WITH 18% ELONGATION IN 8 INCHES.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION.
FIXED ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. FIXED ENDS.
RATIO l/r
fd IN POUNDS PER SQUARE INCH.

FIG. 26.

These experiments by Mr. Christie are the only series on fixed-ended columns of which the writer is aware. It is to be noted that Mr. Christie's Test No. 174, with ratio $\frac{l}{r} = 118$ (maximum load imposed being 24 050 lbs. per square inch), is not shown on the diagram, as failure did not take place.

Wrought Iron.—Flat Ends.

Figs. 27, 28, 29, 30 and 31.—Tests of 240 flat-ended wrought-iron columns. The diagrams are self-explanatory, as far as possible. Attention is directed to the low results evidenced in these tests of flat-ended columns when of considerable length, owing to their rotating on their ends. This mode of failure was found by Mr. Christie in his tests of flat-ended struts, as always occurring in the longest struts, and never in the shortest.

Fig. 32.—Results of seventy-nine tests of flat-ended wrought-iron columns of large size, of various sections, and by various experimenters. The writer has not been able to refer to the original records in every case, but the sources from which the information has been obtained are acknowledged on the diagram.

With regard to the Keystone columns tested by Mr. Bouscaren, only the four which were riveted through the projecting flanges, similarly to a Phœnix column, are recorded on the diagram.

Fig. 33.—Thirteen tests of wrought-iron flat-ended columns, by Professor Bauschinger.* These columns, which had flat ends, are not open to the objections raised against the use, as a basis of reference, of Bauschinger's pivot-ended columns.

Seven results, representing thirteen tests of flat-ended wrought-iron columns, by Professor Tetmajer†.

Twenty tests of flat-ended wrought-iron columns, by the late C. A. Marshall, M. Am. Soc. C. E.‡ Reference to Mr. Marshall's tests will be made subsequently.

Forty results = forty-six tests, total on diagram.

Fig. 34.—This diagram is compounded of Figs. 26 to 33, inclusive, and shows the results of 390 experiments (384 results), on wrought-iron columns, 25 of the tests being of fixed-ended columns, and 365 being of flat-ended columns. This diagram shows practi-

* Fünfzehntes Heft, *Mittheilung*, xviii.
† "Tetmajer's Communications."
‡ *Transactions*, Am. Soc. C. E., Vol. xvii

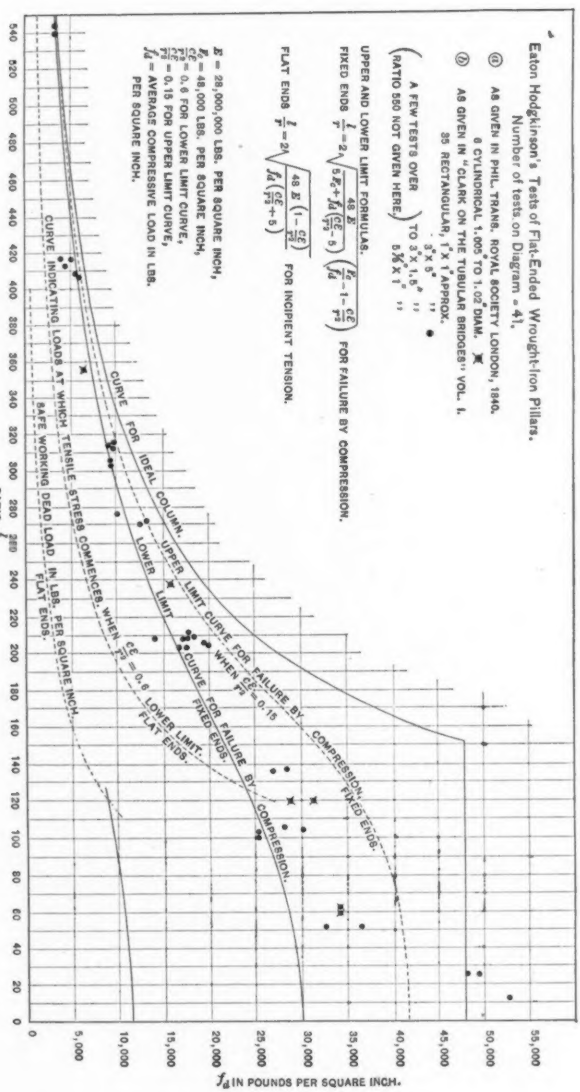

Eaton Hodgkinson's Tests of Flat-Ended Wrought-Iron Pillars.
Number of tests on Diagram = 41.
(a) AS GIVEN IN PHIL. TRANS. ROYAL SOCIETY LONDON, 1840.
6 CYLINDRICAL 1.005" TO 1.02" DIAM.
(b) AS GIVEN IN "CLARK ON THE TUBULAR BRIDGES" VOL. I.
35 RECTANGULAR, 1" X 1" APPROX.
3" X 5"
TO 3" X 1.5"
5⅝" X 1"
A FEW TESTS OVER RATIO 550 NOT GIVEN HERE.
UPPER AND LOWER LIMIT FORMULAS.
FOR FAILURE BY COMPRESSION.
FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH,
E_c = 48,000 LBS. PER SQUARE INCH,
= 0.6 FOR LOWER LIMIT CURVE,
= 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
LOWER LIMIT. FLAT ENDS.
CURVE INDICATING LOADS AT WHICH TENSILE STRESS COMMENCES.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. FLAT ENDS.
RATIO l/r
f_d IN POUNDS PER SQUARE INCH.

FIG. 27

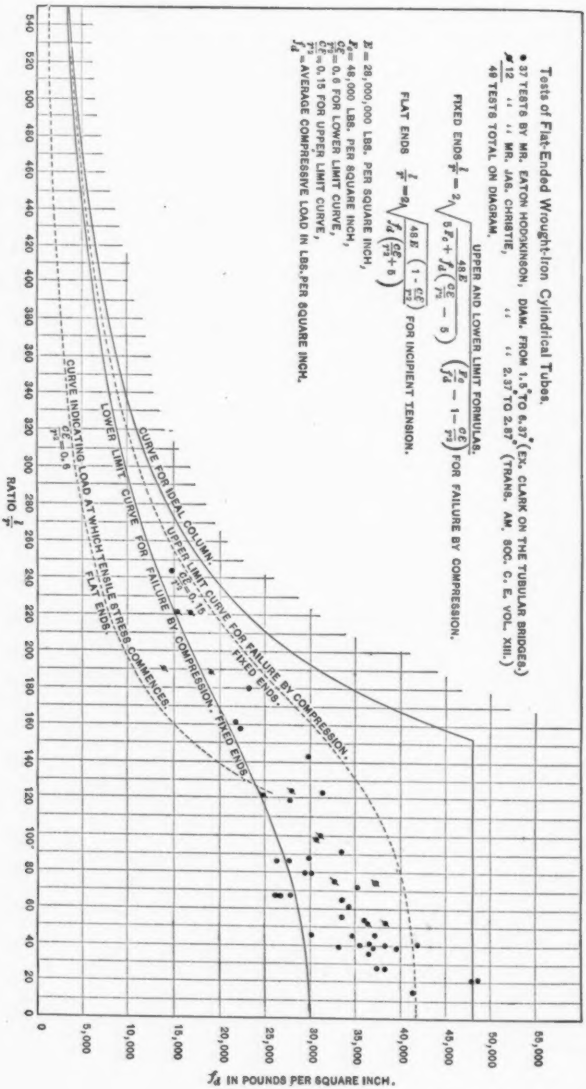

Tests of Flat-Ended Wrought-Iron Cylindrical Tubes.
• 37 TESTS BY MR. EATON HODGKINSON, DIAM. FROM 1.5" TO 6.37" (EX. CLARK ON THE TUBULAR BRIDGES.)
12 " " MR. JAS. CHRISTIE, " " 2.37" TO 2.87" (TRANS. AM. SOC. C. E. VOL. XIII.)
49 TESTS TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS FOR FAILURE BY COMPRESSION.
FLAT ENDS FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = 48,000 LBS. PER SQUARE INCH,
$\frac{c\epsilon}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{c\epsilon}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS. $\frac{c\epsilon}{r^2}$ = 0.15
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. FLAT ENDS. $\frac{c\epsilon}{r^2}$ = 0.6
RATIO $\frac{l}{r}$
f_d IN POUNDS PER SQUARE INCH.

FIG. 28.

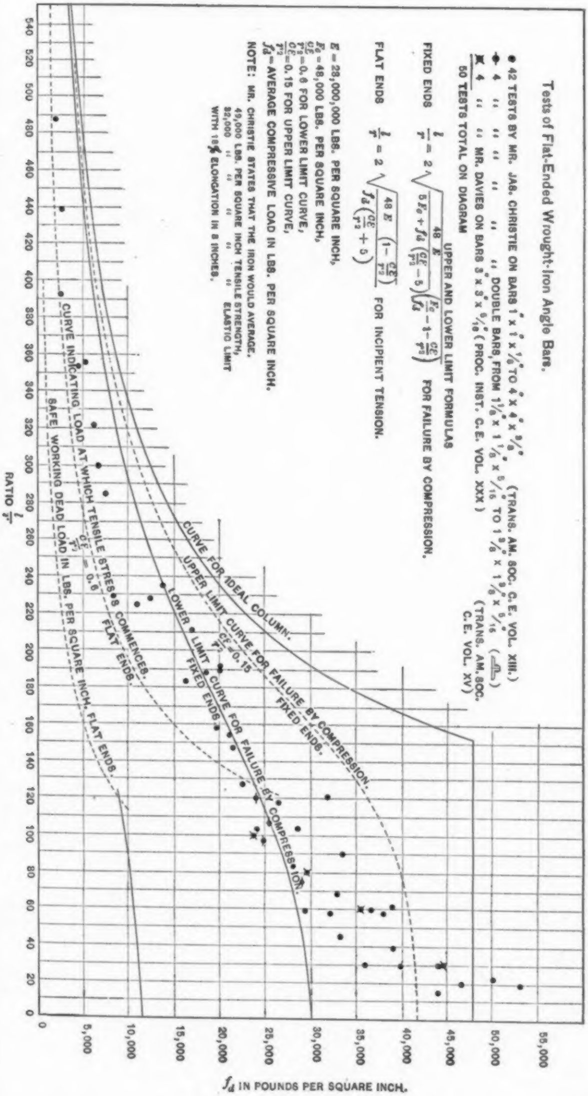
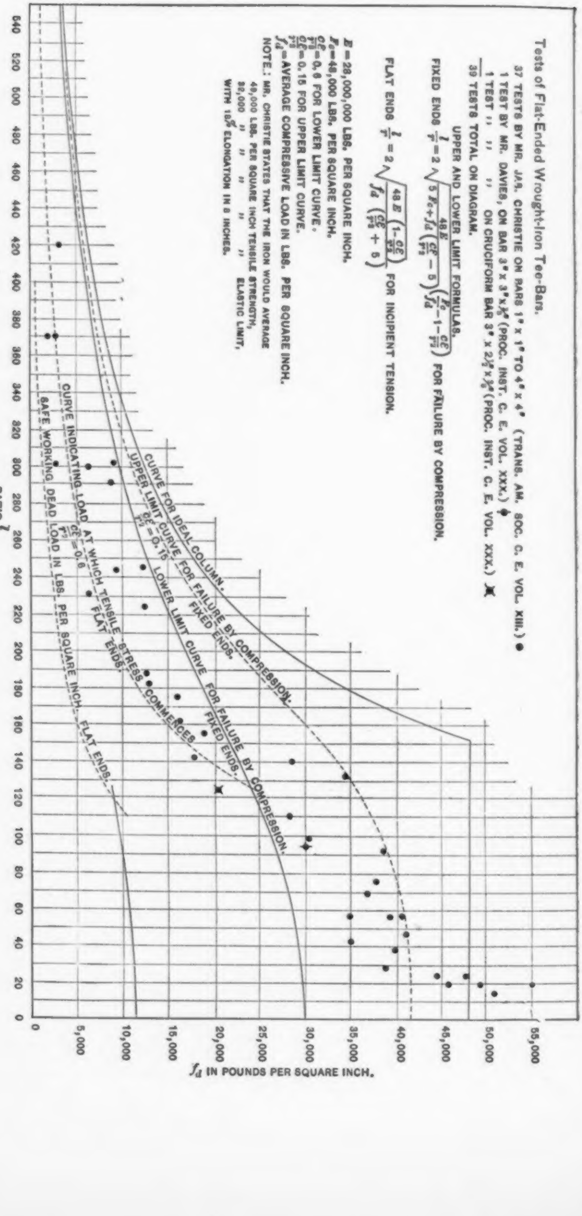

Tests of Flat-Ended Wrought-Iron Angle Bars.
42 TESTS BY MR. JAS. CHRISTIE ON BARS
50 TESTS TOTAL ON DIAGRAM
MR. DAVIES ON BARS (PROC. INST. C.E. VOL. XXX)
(TRANS. AM. SOC. C.E. VOL. XIII.)
(TRANS. AM. SOC. C.E. VOL. XV)
UPPER AND LOWER LIMIT FORMULAS
FIXED ENDS
FOR FAILURE BY COMPRESSION.
FLAT ENDS
FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH.
F_c = 48,000 LBS. PER SQUARE INCH.
$\frac{cE}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{cE}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
NOTE: MR. CHRISTIE STATES THAT THE IRON WOULD AVERAGE,
49,000 LBS. PER SQUARE INCH TENSILE STRENGTH,
32,000 " " " " ELASTIC LIMIT
WITH 18% ELONGATION IN 8 INCHES.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
$\frac{cE}{r^2}$ = 0.15
FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION.
$\frac{cE}{r^2}$ = 0.6
FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES.
FLAT ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. FLAT ENDS.
RATIO $\frac{l}{r}$
f_d IN POUNDS PER SQUARE INCH.

FIG. 29.



Tests of Flat-Ended Wrought-Iron Tee-Bars.
37 TESTS BY MR. JAS. CHRISTIE ON BARS 1" x 1" TO 4" x 4" (TRANS. AM. SOC. C. E. VOL. XIII.)
1 TEST BY MR. DAVIES, ON BAR 3" x 3" x 3/8" (PROC. INST. C. E. VOL. XXX.)
1 TEST " " ON CRUCIFORM BAR 3" x 2 1/2" x 3/8" (PROC. INST. C. E. VOL. XXX.)
39 TESTS TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{ce}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{ce}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS $\frac{l}{r} = 2\sqrt{\frac{48E\left(1 - \frac{ce}{r^2}\right)}{f_d\left(\frac{ce}{r^2} + 5\right)}}$ FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH.
F_c = 48,000 LBS. PER SQUARE INCH.
$\frac{ce}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE.
$\frac{ce}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE.
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
NOTE: MR. CHRISTIE STATES THAT THE IRON WOULD AVERAGE
48,000 LBS. PER SQUARE INCH TENSILE STRENGTH,
32,000 " " " " ELASTIC LIMIT,
WITH 18% ELONGATION IN 8 INCHES.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. $\frac{ce}{r^2}$ = 0.15 FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. $\frac{ce}{r^2}$ = 0.6 FLAT ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. FLAT ENDS.
0 20 40 60 80 100 120 140 160 180 200 220 240 260 280 300 320 340 360 380 400 420 440 460 480 500 520 540
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000 55,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 30.

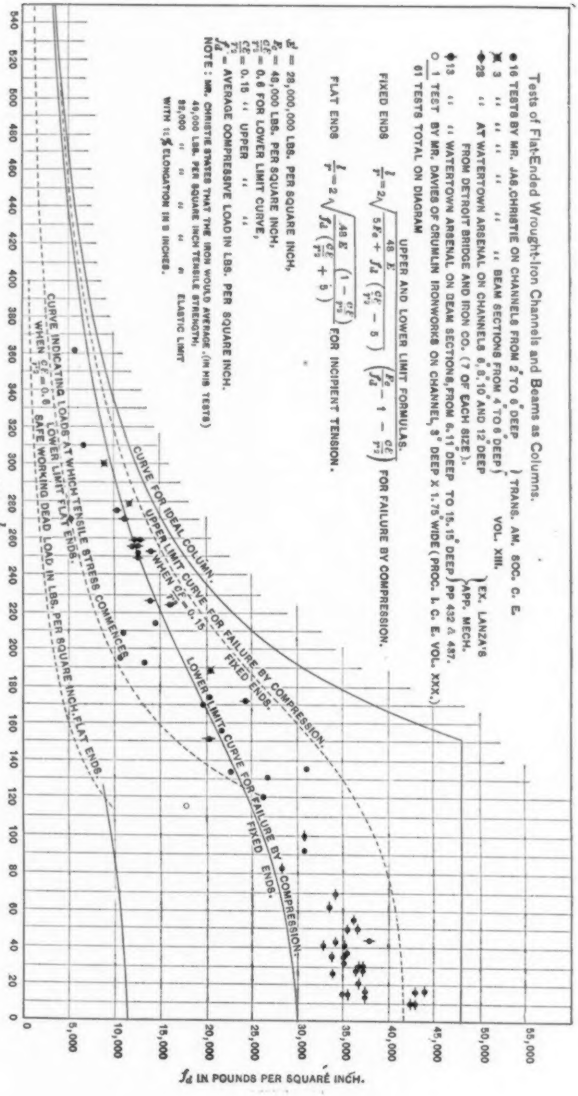

Tests of Flat-Ended Wrought-Iron Channels and Beams as Columns.
● 16 TESTS BY MR. JAS. CHRISTIE ON CHANNELS FROM 2" TO 6" DEEP
3 " " " " BEAM SECTIONS FROM 4" TO 6" DEEP
TRANS. AM. SOC. C. E. VOL. XIII.
28 " AT WATERTOWN ARSENAL ON CHANNELS 6", 8", 10" AND 12" DEEP FROM DETROIT BRIDGE AND IRON CO. (7 OF EACH SIZE).
13 " " WATERTOWN ARSENAL ON BEAM SECTIONS, FROM 6.11" DEEP TO 15.15" DEEP
EX. LANZA'S APP. MECH. PP 432 & 437.
○ 1 TEST BY MR. DAVIES OF CRUMLIN IRONWORKS ON CHANNEL, 8" DEEP X 1.75" WIDE (PROC. I. C. E. VOL. XXX.)
61 TESTS TOTAL ON DIAGRAM
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r}=2\sqrt{\frac{48\ E}{5F_e+f_d\left(\frac{c\varepsilon}{r^2}-5\right)}\left(\frac{F_e}{f_d}-1-\frac{c\varepsilon}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS $\frac{l}{r}=2\sqrt{\frac{48\ E\left(1-\frac{c\varepsilon}{r^2}\right)}{f_d\left(\frac{c\varepsilon}{r^2}+5\right)}}$ FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_e = 48,000 LBS. PER SQUARE INCH,
$\frac{c\varepsilon}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{c\varepsilon}{r^2}$ = 0.15 " UPPER "
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
NOTE: MR. CHRISTIE STATES THAT THE IRON WOULD AVERAGE, (IN HIS TESTS)
49,000 LBS. PER SQUARE INCH TENSILE STRENGTH,
32,000 " " " " ELASTIC LIMIT
WITH 11% ELONGATION IN 8 INCHES.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
WHEN $\frac{c\varepsilon}{r^2}$ = 0.15
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOADS AT WHICH TENSILE STRESS COMMENCES
WHEN $\frac{c\varepsilon}{r^2}$ = 0.6 LOWER LIMIT FLAT ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH, FLAT ENDS.
540 520 500 480 460 440 420 400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000 55,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 81

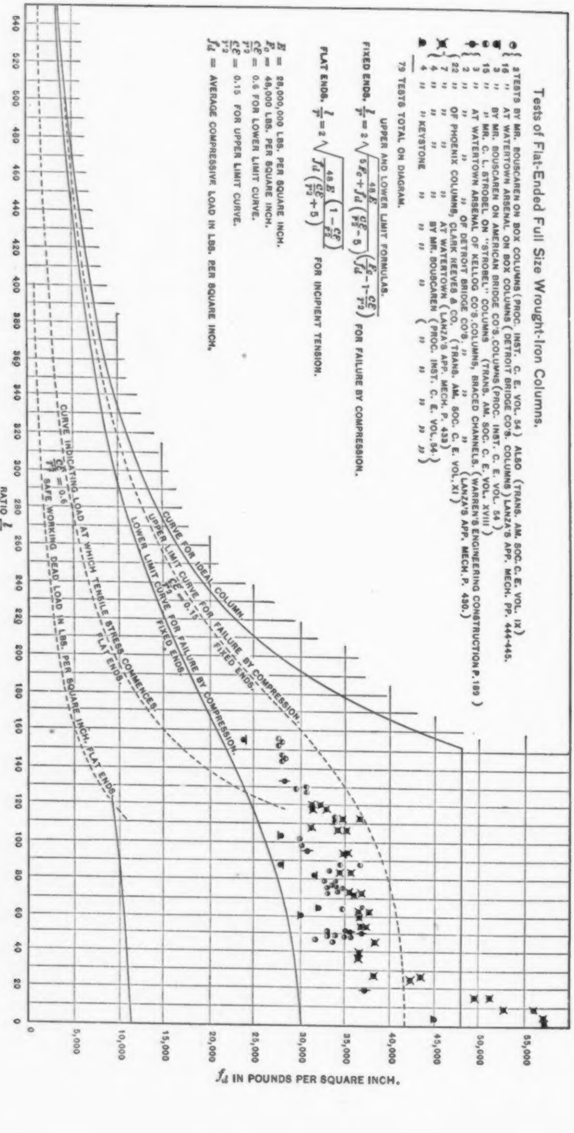

Tests of Flat-Ended Full Size Wrought-Iron Columns.
3 TESTS BY MR. BOUSCAREN ON BOX COLUMNS (PROC. INST. C. E. VOL. 54) ALSO (TRANS. AM. SOC. C. E. VOL. IX)
16 " AT WATERTOWN ARSENAL ON BOX COLUMNS (DETROIT BRIDGE CO'S. COLUMNS) LANZA'S APP. MECH. PP. 444-445.
3 " BY MR. BOUSCAREN ON AMERICAN BRIDGE CO'S. COLUMNS (PROC. INST. C. E. VOL. 54)
15 " " MR. C. L. STROBEL ON "STROBEL" COLUMNS (TRANS. AM. SOC. C. E. VOL. XVIII)
3 " AT WATERTOWN ARSENAL OF KELLOG CO'S. COLUMNS, BRACED CHANNELS. (WARREN'S ENGINEERING CONSTRUCTION P. 189)
2 " " " OF DETROIT BRIDGE CO'S. " " (LANZA'S APP. MECH. P. 430.)
22 " OF PHOENIX COLUMNS, CLARK REEVES & CO. (TRANS. AM. SOC. C. E. VOL. XI)
7 " " " AT WATERTOWN (LANZA'S APP. MECH. P. 439)
4 " " " BY MR. BOUSCAREN (PROC. INST. C. E. VOL. 54.)
4 " " KEYSTONE " " " (" " " " " ")
79 TESTS TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS. $\frac{l}{r} = 2\sqrt{\frac{48\,E}{5F_c + f_d\left(\frac{c\mathcal{E}}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\mathcal{E}}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS. $\frac{l}{r} = 2\sqrt{\frac{48\,E\left(1 - \frac{c\mathcal{E}}{r^2}\right)}{f_d\left(\frac{c\mathcal{E}}{r^2} + 5\right)}}$ FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH.
F_c = 48,000 LBS. PER SQUARE INCH.
$\frac{c\mathcal{E}}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE.
$\frac{c\mathcal{E}}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE.
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. $\frac{c\mathcal{E}}{r^2} = 0.15$ FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. $\frac{c\mathcal{E}}{r^2} = 0.6$ FLAT ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. FLAT ENDS.
RATIO $\frac{l}{r}$
0 20 40 60 80 100 120 140 160 180 200 220 240 260 280 300 320 340 360 380 400 420 440 460 480 500 520 540
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000 55,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 32.

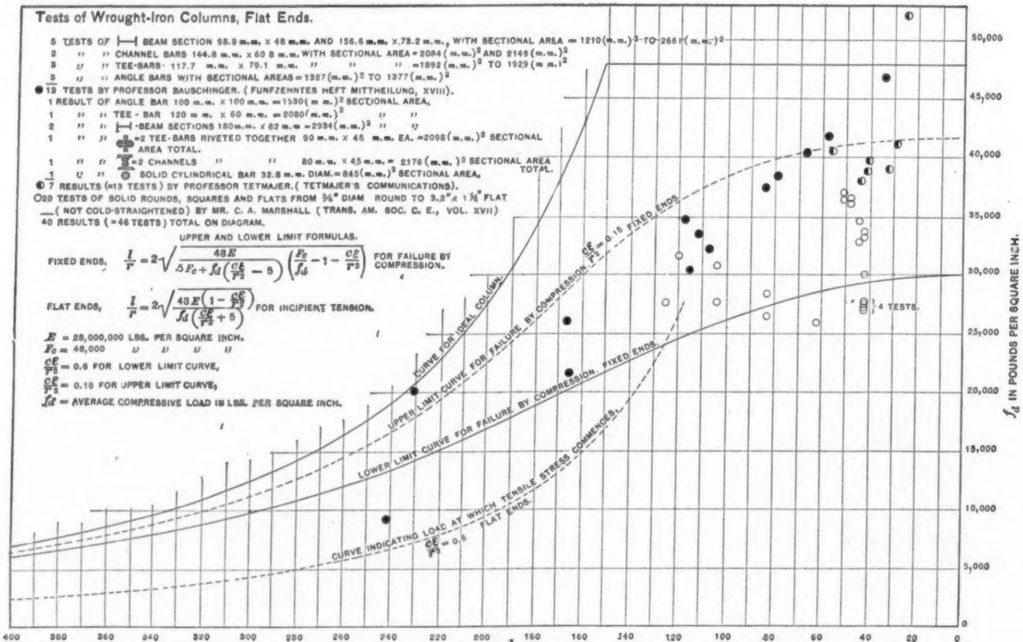

Tests of Wrought-Iron Columns, Flat Ends.
5 TESTS OF BEAM SECTION 98.9 m.m. x 48 m.m. AND 156.6 m.m. x 73.2 m.m., WITH SECTIONAL AREA = 1210(m.m.)² TO 2661(m.m.)²
2 " " CHANNEL BARS 144.8 m.m. x 60 8 m.m. WITH SECTIONAL AREA = 2084 (m.m.)² AND 2149 (m.m.)²
3 " " TEE-BARS 117.7 m.m. x 79.1 m.m. " " " = 1892 (m.m.)² TO 1929 (m.m.)²
3 " " ANGLE BARS WITH SECTIONAL AREAS = 1327 (m.m.)² TO 1377 (m.m.)²
13 TESTS BY PROFESSOR BAUSCHINGER. (FUNFZEHNTES HEFT MITTHEILUNG, XVIII).
1 RESULT OF ANGLE BAR 100 m.m. x 100 m.m. = 1530 (m.m.)² SECTIONAL AREA.
1 " " TEE-BAR 120 m.m. x 60 m.m. = 2080 (m.m.)² " "
2 " " -BEAM SECTIONS 180 m.m. x 82 m.m. = 2934 (m.m.)² " "
1 " " = 2 TEE-BARS RIVETED TOGETHER 90 m.m. x 45 m.m. EA. = 2098 (m.m.)² SECTIONAL AREA TOTAL.
1 " " = 2 CHANNELS " " 80 m.m. x 45 m.m. = 2176 (m.m.)² SECTIONAL AREA TOTAL.
1 " " SOLID CYLINDRICAL BAR 32.8 m.m. DIAM. = 845 (m.m.)² SECTIONAL AREA.
7 RESULTS (= 13 TESTS) BY PROFESSOR TETMAJER. (TETMAJER'S COMMUNICATIONS).
20 TESTS OF SOLID ROUNDS, SQUARES AND FLATS FROM ¾" DIAM ROUND TO 3.2" x 1¾" FLAT (NOT COLD-STRAIGHTENED) BY MR. C. A. MARSHALL (TRANS. AM. SOC. C. E., VOL. XVII)
40 RESULTS (= 46 TESTS) TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS, $\frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{c\varepsilon}{r^2} - 5\right)\left(\frac{F_c}{f_d} - 1 - \frac{c\varepsilon}{r^2}\right)}}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS, $\frac{l}{r} = 2\sqrt{\frac{48E\left(1 - \frac{c\varepsilon}{r^2}\right)}{f_d\left(\frac{c\varepsilon}{r^2} + 5\right)}}$ FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH.
F_c = 48,000 " " " "
$\frac{c\varepsilon}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{c\varepsilon}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. $\frac{c\varepsilon}{r^2}$ = 0.15 FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. $\frac{c\varepsilon}{r^2}$ = 0.6 FLAT ENDS.
4 TESTS.
400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 33.

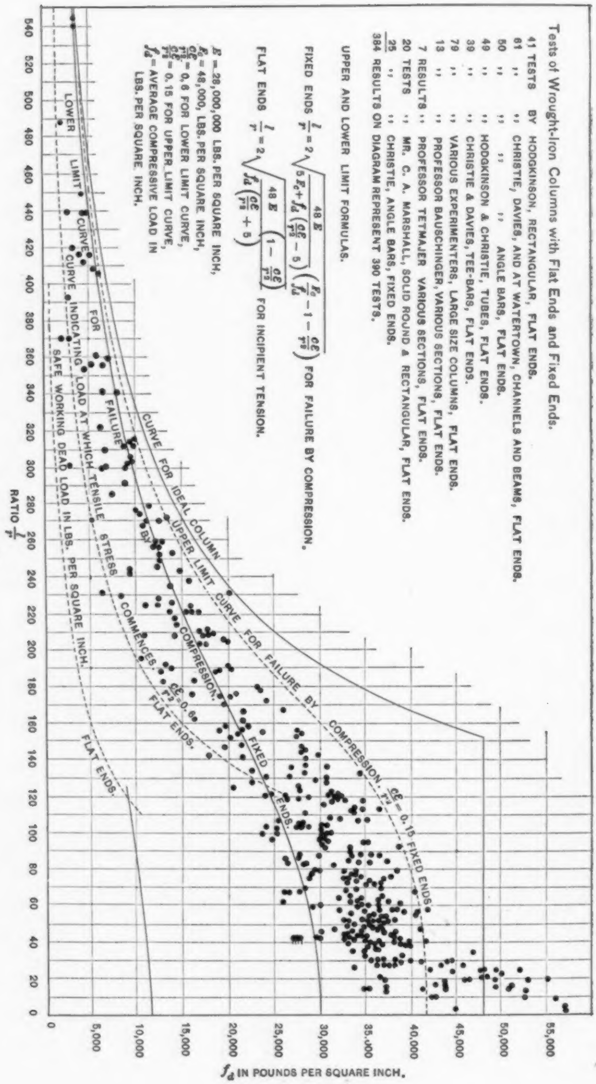

Tests of Wrought-Iron Columns with Flat Ends and Fixed Ends.
41 TESTS BY HODGKINSON, RECTANGULAR, FLAT ENDS.
61 " " CHRISTIE, DAVIES, AND AT WATERTOWN, CHANNELS AND BEAMS, FLAT ENDS.
50 " " " " ANGLE BARS, FLAT ENDS.
49 " " HODGKINSON & CHRISTIE, TUBES, FLAT ENDS.
39 " " CHRISTIE & DAVIES, TEE-BARS, FLAT ENDS.
79 " " VARIOUS EXPERIMENTERS, LARGE SIZE COLUMNS, FLAT ENDS.
13 " " PROFESSOR BAUSCHINGER, VARIOUS SECTIONS, FLAT ENDS.
7 RESULTS " PROFESSOR TETMAJER VARIOUS SECTIONS, FLAT ENDS.
20 TESTS " MR. C. A. MARSHALL, SOLID ROUND & RECTANGULAR, FLAT ENDS.
25 " " CHRISTIE, ANGLE BARS, FIXED ENDS.
384 RESULTS ON DIAGRAM REPRESENT 390 TESTS.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48\,E}{5\,E_c + f_d\left(\frac{ce}{r^2} - 5\right)}\left(\frac{E_c}{f_d} - 1 - \frac{ce}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS $\frac{l}{r} = 2\sqrt{\frac{48\,E\left(1 - \frac{ce}{r^2}\right)}{f_d\left(\frac{ce}{r^2} + 5\right)}}$ FOR INCIPIENT TENSION.
E = 28,000,000 LBS. PER SQUARE INCH,
E_c = 48,000, LBS. PER SQUARE INCH,
$\frac{ce}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{ce}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION, $\frac{ce}{r^2}$ = 0.15 FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. $\frac{ce}{r^2}$ = 0.6 FLAT ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCH. FLAT ENDS.
540 520 500 480 460 440 420 400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000 55,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 34.

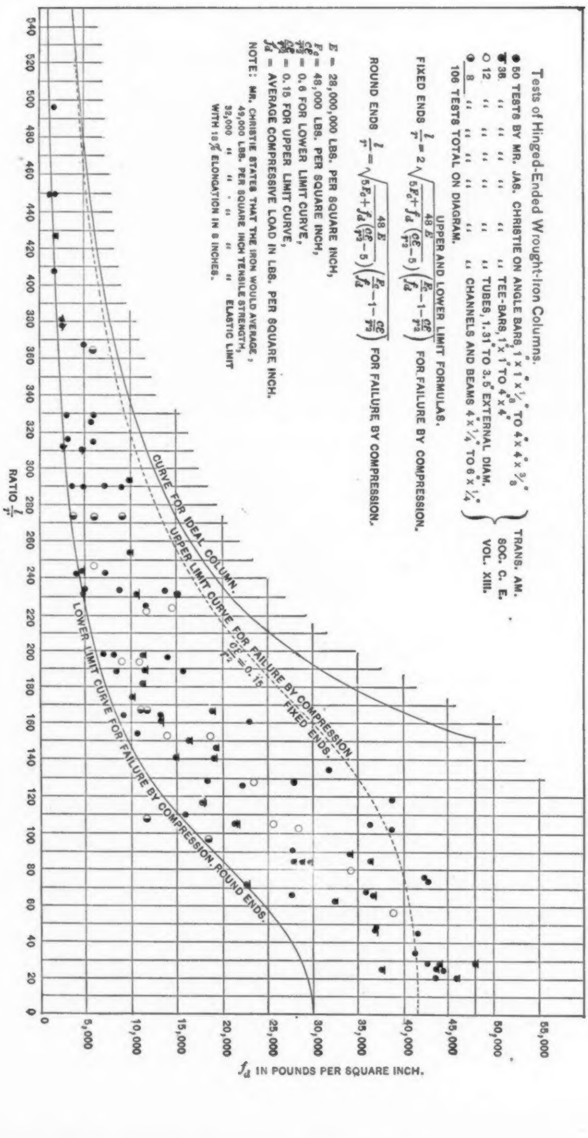

Tests of Hinged-Ended Wrought-Iron Columns.
● 50 TESTS BY MR. JAS. CHRISTIE ON ANGLE BARS 1"x1"x1/8" TO 4"x4"x3/8"
38 " " " " " TEE-BARS, 1"x1" TO 4"x4"
○ 12 " " " " " TUBES, 1.31" TO 3.5" EXTERNAL DIAM.
8 " " " " " CHANNELS AND BEAMS 4"x1/4" TO 6"x1/4"
106 TESTS TOTAL ON DIAGRAM.
TRANS. AM. SOC. C. E. VOL. XIII.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48\,E}{5F_e + f_d\left(\frac{cc}{r^2}-5\right)}\left(\frac{F_e}{f_d}-1-\frac{cc}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48\,E}{5F_e + f_d\left(\frac{cc}{r^2}-5\right)}\left(\frac{F_e}{f_d}-1-\frac{cc}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_e = 48,000 LBS. PER SQUARE INCH,
$\frac{cc}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{cc}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
NOTE: MR. CHRISTIE STATES THAT THE IRON WOULD AVERAGE ;
49,000 LBS. PER SQUARE INCH TENSILE STRENGTH,
32,000 " " " " ELASTIC LIMIT
WITH 18% ELONGATION IN 8 INCHES.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION, $\frac{cc}{r^2}$ = 0.15 FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION, ROUND ENDS.
RATIO $\frac{l}{r}$
f_d IN POUNDS PER SQUARE INCH.
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000 55,000
0 20 40 60 80 100 120 140 160 180 200 220 240 260 280 300 320 340 360 380 400 420 440 460 480 500 520 540

FIG. 85

cally all the available experimental data on the strength of wrought-iron columns with flat ends.

The weakness of flat-ended columns in the higher ratios of $\frac{l}{r}$ is most clearly evident in this diagram. The high strength shown by a number of columns of ratio $\frac{l}{r} = 30$, and under, is also a striking feature, and it would appear that columns longer than 30 radii of gyration ($7\frac{1}{2}$ diameters in cylindrical solid bars) cannot be expected to develop higher strength due to plastic yielding under compression, and that column action begins to come into play more definitely at about ratios $\frac{l}{r} = 30$ to 40.

It is evident that in very short columns the useful ultimate strength must be measured by the elastic limit under compression, and this may not have been noted carefully in some of the experiments, but beyond the ratio 30, the influence of column length appears to become very strongly marked.

Wrought Iron.—Hinged or Pin Ends.

Fig. 35.—One hundred and six tests of hinged-ended angles, tees, tubes, channels and beams, by Mr. Christie.* In Mr. Christie's account of his tests, he remarks: "The hinged-ended tests varied all the way from the value of round-ended up to flat-ended."

The writer would extend this to, "the hinged-ended tests varied all the way from the lower values for round-ended up to the higher values for flat or fixed-ended."

The truth of this is apparent on referring to the diagram on which the writer has plotted the lower-limit curve for round-ended columns with $\frac{c\,\varepsilon}{r^2} = 0.6$, as on Figs. 20, 21 and 22, and the upper-limit dotted curve for fixed-ended columns with $\frac{c\,\varepsilon}{r^2} = 0.15$, as on Figs. 26 to 33.

The high results for ratios under 30 are again evident, and are similar to the results for flat-ended columns.

The tests on hinged-ended angles, given in Table No. 3 of Mr. Christie's paper, are all plotted on the diagram, excepting those which were not carried to the point of failure. In addition to these, the writer has also plotted five of Mr. Christie's extra tests on some of these angle bars, as given on pages 113 and 114 of Mr. Christie's paper.

***Transactions*, Am. Soc. C. E., Vol. xiii.

These are as follows:

1. First test, with 2-in. balls and sockets, apparently central, on the same bar as in Experiment No. 80, ratio $\frac{l}{r} = 164$, for which the result of the second test after moving the ends 0.06 in. from the original position was given in Mr. Christie's Table No. 3. The first test, apparently central, gave an ultimate strength of 19 400 lbs. (9 200 lbs. per square inch), and the second test, after moving the specimen 0.06 in., gave an ultimate strength of 27 850 lbs. (13 199 lbs. per square inch), Both of these results are on the diagram.

2 and 3. The second and third tests on the same bar as in Experiment No. 117, ratio $\frac{l}{r} = 290$. The first test (No. 117) on this bar was with 2-in. balls and sockets (7 020 lbs. per square inch), the second with 1-in. balls and sockets (3 525 lbs. per square inch), and the third with 2-in. pins. (4 790 lbs. per square inch).

4. The second test on the same bar as in Experiment No. 118, ratio $\frac{l}{r} = 233$. In the case of this bar, the first test result, with ends apparently central, is given in Mr. Christie's Table No. 3, the strength being 16 700 lbs. (8 650 lbs. per square inch), and in the second test the ends were moved 0.06 in., with the result that the strength was 26 450 lbs. (13 700 lbs. per square inch).

5. The first test on the same bar as in Experiment No. 119, ratio $\frac{l}{r} = 189$, giving a strength of 20 150 lbs. (8 340 lbs. per square inch). In the case of this bar, the second test, after moving the bar 0.07 in., with resulting ultimate strength of 38 175 lbs. (15 770 lbs. per square inch), is that given in Mr. Christie's Table No. 3.

It is not clear why Mr. Christie should have selected the second tests in the case of Experiments Nos. 80 and 119, for insertion in his Table No. 3, which otherwise referred to struts under apparently central loads.

Mr. Christie's extra tests, of considerable number, on angles, tees, tubes, and H-beams, in addition to the few above mentioned, made it most apparent that the physical axis did not always coincide with the geometric axis, and they also showed very markedly how very sensitive the columns were to apparently insignificant amounts of adjustment, and to variations in sizes of pin or socket.

Fig. 36.—One hundred and twenty-six tests of pin-ended wrought-iron columns of large size by various experimenters. The writer has not had the advantage of referring to the original published records of most of the tests, and he has therefore been compelled to take the records at second hand from the sources acknowledged on the diagram.

With regard to the Detroit Bridge Company's columns, the results have been taken from the paper by Theodore Cooper, M. Am. Soc. C. E.,* from Professor Lanza's "Applied Mechanics," and Professor W. H. Warren's "Engineering Construction."

The results, in many cases, if plotted to the values of $\frac{l}{r}$ given by Mr. Cooper, would differ considerably from the results as plotted on the diagram. This results from the writer's having found that the values of the ratio $\frac{l}{r}$, given by Mr. Cooper, do not agree with the dimensions of the columns as given by Professor Lanza and Professor Warren. The writer, accordingly, recalculated the values of $\frac{l}{r}$, using the lengths as given by Professors Lanza and Warren, and, calculating the radius of gyration from the detailed sections given on Plate XXVII of Mr. Cooper's paper, at the same time comparing carefully the sections as given by Mr. Cooper with the less fully detailed sections given by Professors Lanza and Warren.

It would appear that Mr. Cooper has taken the least radius of gyration in every case, but these columns were pin-ended, in which the radius of gyration at right angles to the pin was frequently greater than that parallel to the pin, and it is this greater radius of gyration which should be used, as the columns were practically flat-ended as regards failure in the direction of the least radius of gyration. The effect of this is that Mr. Cooper's values of $\frac{l}{r}$ are in many cases too great, and, consequently, too great strength has been credited to these high values of $\frac{l}{r}$.

It is worthy of note that a few of these columns did fail by deflecting in a direction parallel to the pins, and acting as flat-ended columns, and not as pin-ended. This draws attention to the necessity for care in proportioning pin-ended columns in which the radius of

* "American Railway Bridges," *Transactions*, Am. Soc. C. E., Vol. xxi.

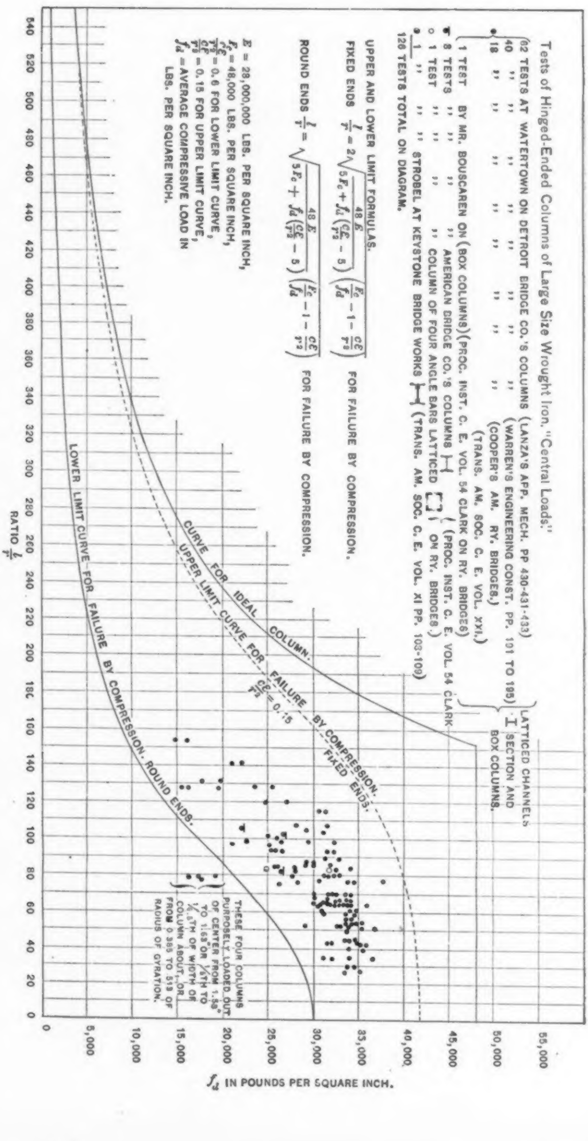
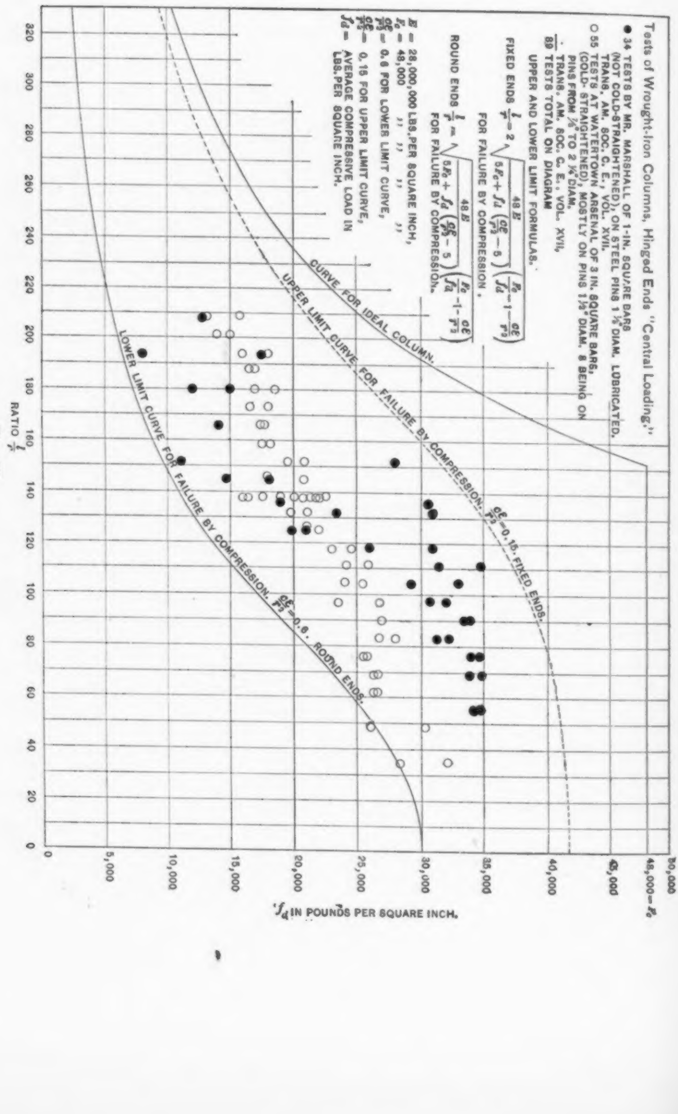
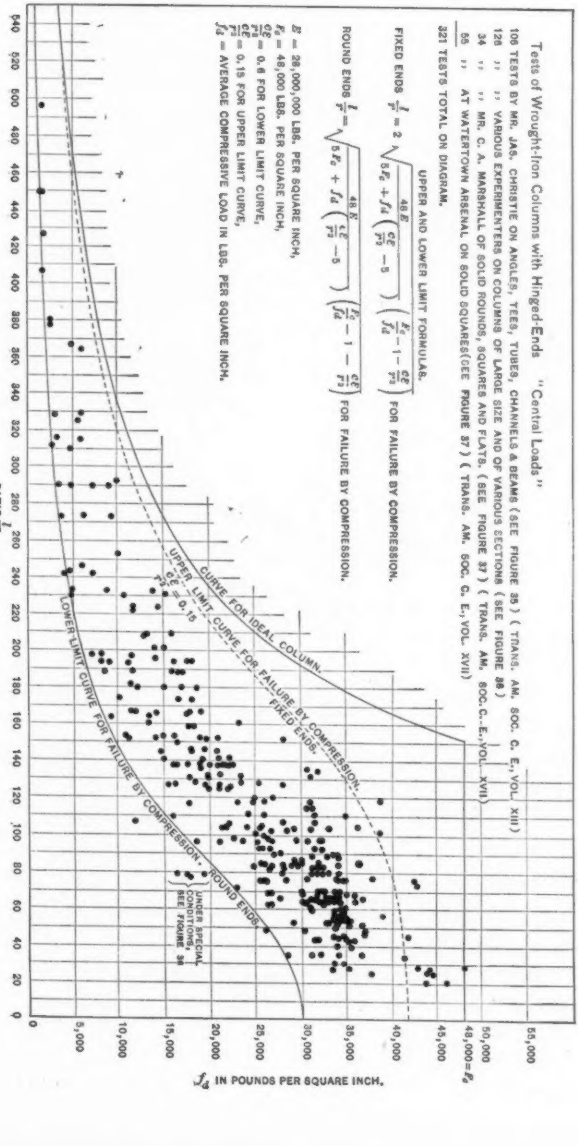

Tests of Hinged-Ended Columns of Large Size Wrought Iron. "Central Loads."
62 TESTS AT WATERTOWN ON DETROIT BRIDGE CO.'S COLUMNS (LANZA'S APP. MECH. PP 430-431-433)
40 " " " " " " " " (WARREN'S ENGINEERING CONST. PP. 191 TO 195)
18 " " " " " " " " (COOPER'S AM. RY. BRIDGES.)
(TRANS. AM. SOC. C. E. VOL. XXI.)
LATTICED CHANNELS I SECTION AND BOX COLUMNS.
1 TEST BY MR. BOUSCAREN ON (BOX COLUMNS)(PROC. INST. C. E. VOL. 54 CLARK ON RY. BRIDGES)
8 TESTS " " " " AMERICAN BRIDGE CO.'S COLUMNS
1 TEST " " " " COLUMN OF FOUR ANGLE BARS LATTICED
(PROC. INST. C. E. VOL. 54 CLARK ON RY. BRIDGES.)
1 " " " STROBEL AT KEYSTONE BRIDGE WORKS (TRANS. AM. SOC. C. E. VOL. XI PP. 103-109)
126 TESTS TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{c\mathcal{E}}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\mathcal{E}}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48E}{5F_c + f_d\left(\frac{c\mathcal{E}}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\mathcal{E}}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = 48,000 LBS. PER SQUARE INCH,
$\frac{c\mathcal{E}}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{c\mathcal{E}}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
$\frac{c\mathcal{E}}{r^2} = 0.15$
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. ROUND ENDS.
THESE FOUR COLUMNS PURPOSELY LOADED OUT OF CENTER FROM 1.59" TO 1.63" OR 1/5TH TO 1/6.5TH OF WIDTH OF COLUMN ABOUT, OR FROM 0.385 TO .513 OF RADIUS OF GYRATION.
540 520 500 480 460 440 420 400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 50,000 55,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 36.



Tests of Wrought-Iron Columns, Hinged Ends "Central Loading."
● 34 TESTS BY MR. MARSHALL OF 1-IN. SQUARE BARS (NOT COLD-STRAIGHTENED), ON STEEL PINS 1½" DIAM. LUBRICATED. TRANS. AM. SOC. C. E., VOL. XVII.
○ 55 TESTS AT WATERTOWN ARSENAL OF 3 IN. SQUARE BARS. (COLD-STRAIGHTENED), MOSTLY ON PINS 1½" DIAM. 8 BEING ON PINS FROM ¾" TO 2¼" DIAM. TRANS. AM. SOC. C. E., VOL. XVII.
89 TESTS TOTAL ON DIAGRAM
UPPER AND LOWER LIMIT FORMULAS.
FOR FAILURE BY COMPRESSION.
FOR FAILURE BY COMPRESSION.
E = 28,000,000 LBS. PER SQUARE INCH,
AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION.
IN POUNDS PER SQUARE INCH.

FIG. 37.



Tests of Wrought-Iron Columns with Hinged-Ends "Central Loads"
106 TESTS BY MR. JAS. CHRISTIE ON ANGLES, TEES, TUBES, CHANNELS & BEAMS (SEE FIGURE 35) (TRANS. AM. SOC. C. E., VOL. XIII)
126 " " VARIOUS EXPERIMENTERS ON COLUMNS OF LARGE SIZE AND OF VARIOUS SECTIONS (SEE FIGURE 36)
34 " " MR. C. A. MARSHALL OF SOLID ROUNDS, SQUARES AND FLATS. (SEE FIGURE 37) (TRANS. AM. SOC. C. E., VOL. XVII)
55 " AT WATERTOWN ARSENAL ON SOLID SQUARES (SEE FIGURE 37) (TRANS. AM. SOC. C. E., VOL. XVII)
321 TESTS TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{ce}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{ce}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
ROUND ENDS $\frac{l}{r} = \sqrt{\frac{48E}{5F_c + f_d\left(\frac{ce}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{ce}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
E = 28,000,000 LBS. PER SQUARE INCH,
F_c = 48,000 LBS. PER SQUARE INCH,
$\frac{ce}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{ce}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
$\frac{ce}{r^2} = 0.15$
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. ROUND ENDS.
UNDER SPECIAL CONDITIONS, SEE FIGURE 38
540 520 500 480 460 440 420 400 380 360 340 320 300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$
0 5,000 10,000 15,000 20,000 25,000 30,000 35,000 40,000 45,000 48,000 = F_c 50,000 55,000
f_d IN POUNDS PER SQUARE INCH.

FIG. 38.

gyration, parallel to the axis of the pins, is of less value than the radius perpendicular to the pins.

If too great a difference exists, the column may be weaker as a flat-ended column deflecting parallel to the pins than as a pin-ended column deflecting at right angles to the pins.

Four of the Detroit Bridge Company's columns were purposely loaded out of axis, and these are plotted on the diagram with a note to that effect. Two of the columns were formed of a pair of 10-in.

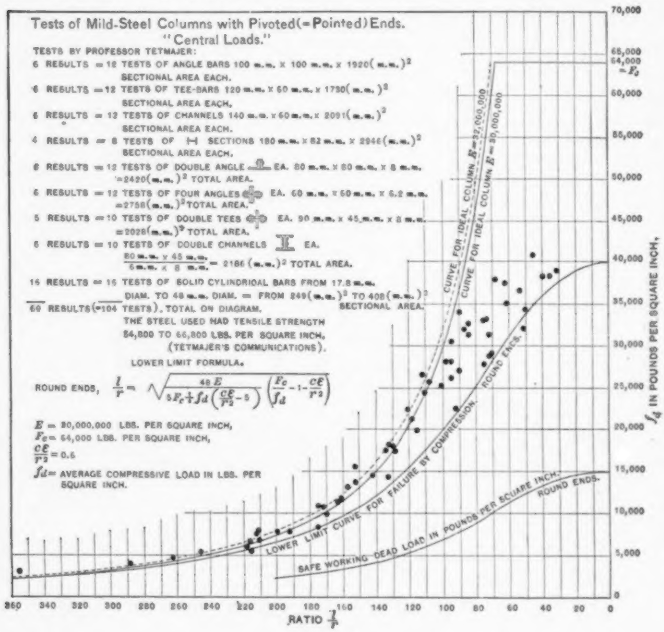


FIG. 39.

channels connected on one side by a plate, and on the other by latticing. The other two were of the same type, but with 8-in. channels.

The low strengths are worthy of careful attention, as the results were evidently not entirely due to these four columns being of unsymmetrical section, but to the pins not being placed at the center of area, since the highest result shown on the whole diagram was of the same type and proportions as these four columns, but with pins placed at the center of area.

Fig. 37.—This diagram represents:

Thirty-four tests of pin-ended wrought-iron columns by Mr. C. A. Marshall;* and fifty-five tests of pin-ended wrought-iron columns at Watertown;† a total of eighty-nine tests. A reference to these will be made when dealing with Mr. Marshall's tests on pin-ended steel columns.

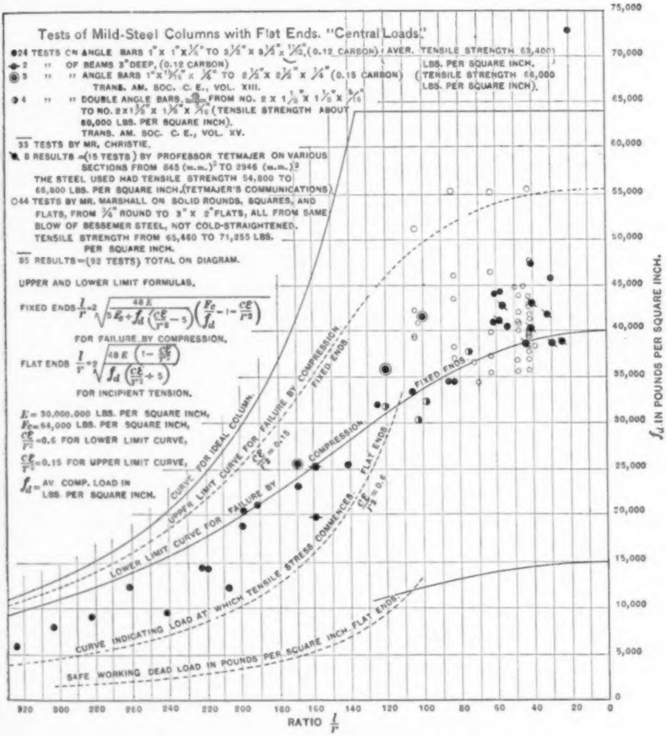


FIG. 40.

Fig. 38.—This is a combination of the diagrams on Figs. 35, 36 and 37, and shows the results of 321 tests of hinged-ended wrought-iron columns.

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* *Transactions*, Am. Soc. C. E., Vol. xvii.

† Replotted from Plate XI of Mr. Marshall's paper, *Transactions*, Am. Soc. C. E., Vol. xvii.

Mild Steel.—Pivoted Ends.

Fig. 39.—Sixty results, representing 104 tests by Professor Tetmajer on mild-steel columns of various sections.* The remarks made with regard to Tetmajer's pivot-ended wrought-iron tests, shown on Fig. 21, apply also to these tests on mild steel.

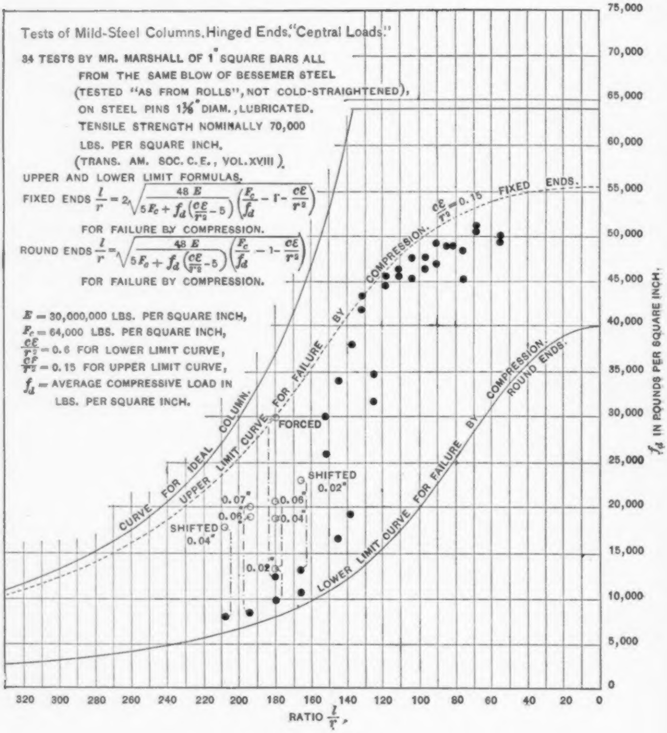

FIG. 41.

Mild Steel.—Flat Ends.

Fig. 40.—This diagram represents:

Twenty-nine tests by Mr. Christie;† four tests by Mr. Christie;‡ eight

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* "Tetmajer's Communications."

† *Transactions*, Am. Soc. C. E., Vol. xiii.

‡ *Transactions*, Am. Soc. C. E., Vol. xv.

results by Professor Tetmajer,* representing fifteen tests; forty-four tests by Mr. Marshall;† a total of eighty-five results, ninety-two tests.

Mild Steel.—Hinged Ends.

Fig. 41.—Thirty-four tests of hinged-ended mild-steel columns by Mr. Marshall.‡ Mr. Marshall stated in his paper that his results indicated clearly the law that:

"The elastic limit of the material is the chief factor in determining the resistance of struts of ordinary length made out of wrought iron or steel, excepting the very hardest kinds, and that the two quantities elastic limit in compression and ultimate compressive strength are identical within a very considerable range of length of columns."

Mr. Marshall's conclusions were supported strongly by the results of his experiments, but the number of tests made was hardly sufficient to warrant the acceptance, in practice, of conclusions, which, if consistently applied, would allow the same stress upon a wrought-iron column having a length of 100 times the radius of gyration, as for one of only a third of that length.

The tests made by other experimenters show a considerable fall in strength with increase of length, and their evidence cannot be ignored in practice; and, judging from the results obtained by Hodgkinson, Christie, and Tetmajer, with pivot-ended columns in which the influence of frictional resistance or fixity of the column ends was eliminated, the range of length within which Mr. Marshall's law would apply, is very limited.

It will be noticed on referring to the diagrams Figs. 37 and 41, that Mr. Marshall's pin-ended tests exhibited a very rapid fall in strength in the longer column, *i. e.*, at the ratios 110 to 120 and over. This points to the probability that columns shorter than, but near, these ratios must have been in a very unstable condition, and were assisted greatly by the frictional resistance of the pin-ends.

It is also to be noted that the pin-ended tests made by Mr. Marshall on wrought iron and steel, were compared by him with the "compressive elastic limit" as determined from flat-ended and hinged-ended specimens 1 in. square by 12 ins. long,§ having a ratio $\frac{l}{r} = 41.5$, a

* "Tetmajer's Communications."

† Table No. 1 of Mr. Marshall's paper, *Transactions*, Am. Soc. C. E., Vol. xvii.

‡ Table No. 8 of Mr. Marshall's paper.

§ Table No. 7 of Mr. Marshall's paper.

column length quite sufficient to bring in influences which might, to a very considerable extent, affect the results, and all of these 12-in. bars deflected under their loads. In a few of the longer columns, Mr. Marshall showed how exceedingly sensitive they were to small adjustments in the testing machine, and some of these are indicated on Fig. 41. A very large increase of strength was obtained by merely moving the specimen a small fraction of an inch out of its original centering, and by the simple application of pressure at the start of a second test of a column having ratio $\frac{l}{r} = 180$, in order to cause failure by bending in the opposite direction to that in which it had first yielded, the strength was raised from 12 420 to 29 810 lbs. per square inch. This sensitiveness undoubtedly holds, in some degree, for shorter columns as well as for longer ones.

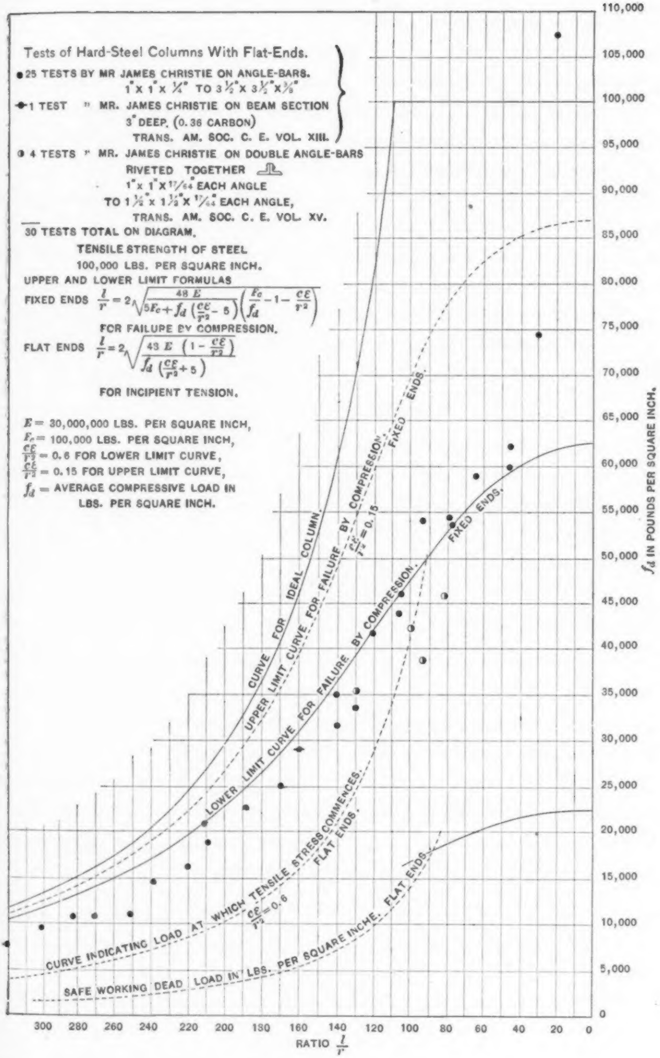
Mr. Marshall's comparison cannot therefore fairly be considered as referred to "compressive elastic limit" of the material, but only as one between the strength of certain columns, 41.5 radii of gyration long, and others of greater length.

The flat-ended tests of steel columns made by Mr. Marshall (Fig. 40) were, however, in part compared with short specimens, two sides long, from the same bars tested as columns, and it is this comparison which lends the greatest strength to Mr. Marshall's conclusions.

It is necessary to keep in view that even in a short compression specimen, eccentricity of loading will have more or less effect, and an ordinary test will only give the average stress on the material, while the important point to be determined as regards column strength is the maximum elastic fiber stress.

Mr. Marshall also made comparison between the ultimate strength of his column and the tensile elastic limit of the same bars as the columns, and the result of the comparison in most cases showed a very remarkable agreement. It is difficult, however, to see how any fixed relation of equality can exist between the elastic limit of a tension specimen and the compressive strength of a column. In a tension test, any inaccuracy of loading or lack of straightness does not tend to increase in influence, while in a compression specimen, even of moderate length, the reverse is the case.

In the curves plotted on Fig. 40 it will be noticed that no attempt is made to differentiate between the soft steel of Professor Tetmajer's



Tests of Hard-Steel Columns With Flat-Ends.
25 TESTS BY MR JAMES CHRISTIE ON ANGLE-BARS. 1" x 1" x 1/4" TO 3 1/2" x 3 1/2" x 3/8"
1 TEST " MR. JAMES CHRISTIE ON BEAM SECTION 3" DEEP. (0.36 CARBON)
TRANS. AM. SOC. C. E. VOL. XIII.
4 TESTS " MR. JAMES CHRISTIE ON DOUBLE ANGLE-BARS RIVETED TOGETHER
1" x 1" x 17/64" EACH ANGLE TO 1 1/2" x 1 1/2" x 17/64" EACH ANGLE,
TRANS. AM. SOC. C. E. VOL. XV.
30 TESTS TOTAL ON DIAGRAM.
TENSILE STRENGTH OF STEEL 100,000 LBS. PER SQUARE INCH.
UPPER AND LOWER LIMIT FORMULAS
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48\,E}{5F_c + f_d\left(\frac{c\varepsilon}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\varepsilon}{r^2}\right)}$
FOR FAILURE BY COMPRESSION.
FLAT ENDS $\frac{l}{r} = 2\sqrt{\frac{48\,E\left(1 - \frac{c\varepsilon}{r^2}\right)}{f_d\left(\frac{c\varepsilon}{r^2} + 5\right)}}$
FOR INCIPIENT TENSION.
E = 30,000,000 LBS. PER SQUARE INCH,
F_c = 100,000 LBS. PER SQUARE INCH,
$\frac{c\varepsilon}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
$\frac{c\varepsilon}{r^2}$ = 0.15 FOR UPPER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION $\frac{c\varepsilon}{r^2} = 0.15$ FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. $\frac{c\varepsilon}{r^2} = 0.6$ FLAT ENDS.
SAFE WORKING DEAD LOAD IN LBS. PER SQUARE INCHE. FLAT ENDS.
110,000
105,000
100,000
95,000
90,000
85,000
80,000
75,000
70,000
65,000
60,000
55,000
50,000
45,000
40,000
35,000
30,000
25,000
20,000
15,000
10,000
5,000
0
f_d IN POUNDS PER SQUARE INCH.
300 280 260 240 220 200 180 160 140 120 100 80 60 40 20 0
RATIO $\frac{l}{r}$

FIG. 42.

tests and the somewhat stronger steel used by Mr. Marshall, with Mr. Christie's standing intermediate.

The number of tests in any of the three sets is not sufficient to warrant any nicety of distinction between the materials used in each, and it may be remarked that the complete range of strength, from Professor Tetmajer's minimum tensile strength of 54 800 lbs. per square inch to Mr. Marshall's tensile maximum of 71 255 lbs. per square inch, is very little greater in proportion than the range of tensile strength found in Professor Tetmajer's tensile tests of one class of wrought iron, in which the minimum and maximum tensile strengths were respectively 46 000 and 59 000 lbs. per square inch.

The curves in Figs. 39, 40 and 41, therefore, have been plotted with a value of $F_c = 64\,000$ lbs., and may be taken as applying to steel of tensile strength of from 60 000 to 70 000 lbs. per square inch.

Hard Steel.—Flat Ends.

Fig. 42.—This diagram represents:

Twenty-six tests by Mr. Christie,* and four tests by Mr. Christie,† a total of thirty tests. It may be urged with regard to this diagram that the lower-limit curve for failure by incipient tension is too low, and is not justified by the experiments.

This curve is not dependent in any way on the ultimate or elastic strength of the material, but solely on the modulus of elasticity and on the value of $\frac{c\,\varepsilon}{r^2}$, and the latter quantity has been taken the same as for all the other diagrams. The evidence of the more numerous tests on wrought iron is so strong that it does not appear advisable to count on higher loads than those indicated.

There is no reason why hard-steel struts should be assumed to be loaded with greater accuracy, or as having greater immunity from injury by cold-straightening, or as being less likely to have initial bends, than those of wrought iron or mild steel.

Timber.—Flat Ends.

Fig. 43. Tests of Yellow Pine or Pitch Pine.—The diagram represents:

Sixty-nine tests, with flat ends, made at Watertown Arsenal.‡ The

* *Transactions*, Am. Soc. C. E., Vol. xiii.

† *Transactions*, Am. Soc. C. E., Vol. xv.

‡ Exec. Doc. 12, 47th Congress, first session. These have been plotted from the figures given in Professor Lanza's "Applied Mechanics," pp. 664 to 668, 5th edition.

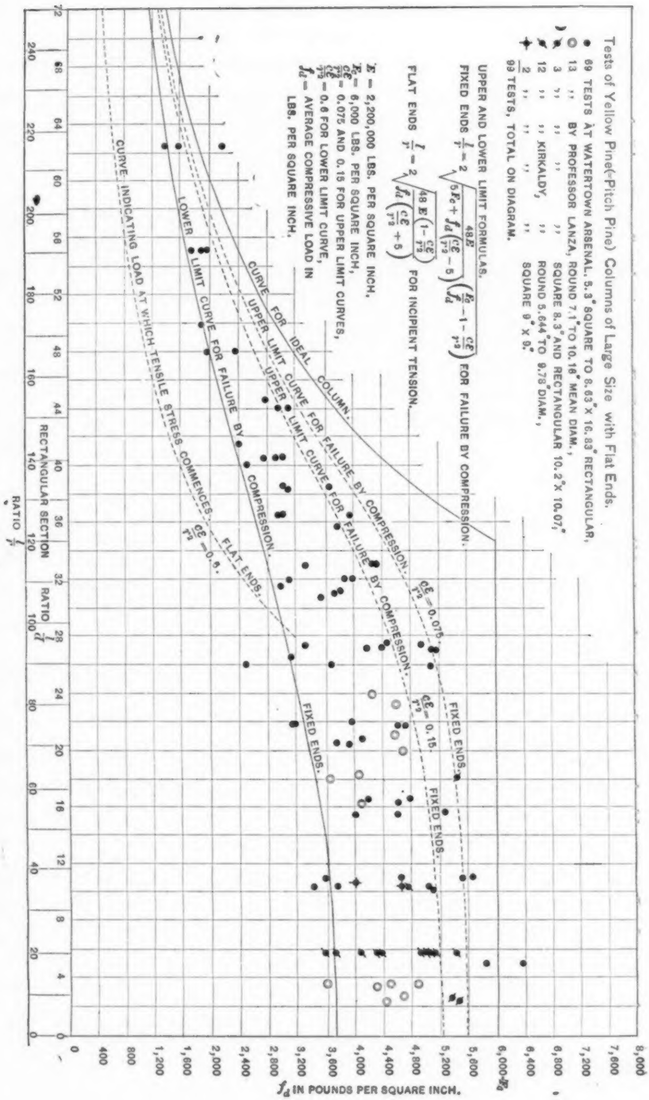

Tests of Yellow Pine (Pitch Pine) Columns of Large Size with Flat Ends.
69 TESTS AT WATERTOWN ARSENAL. 5.3" SQUARE TO 8.63" X 16.63" RECTANGULAR,
13 " BY PROFESSOR LANZA, ROUND 7.1" TO 10.16" MEAN DIAM.,
3 " " " SQUARE 8.3" AND RECTANGULAR 10.2" X 10.07",
12 " " KIRKALDY, " ROUND 5.644" TO 9.78" DIAM.,
2 " " " " SQUARE 9" X 9".
99 TESTS, TOTAL ON DIAGRAM.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS $\frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{cE}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{cE}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS $\frac{l}{r} = 2\sqrt{\frac{48E\left(1 - \frac{cE}{r^2}\right)}{f_d\left(\frac{cE}{r^2} + 5\right)}}$ FOR INCIPIENT TENSION.
E = 2,200,000 LBS. PER SQUARE INCH,
F_c = 6,000 LBS. PER SQUARE INCH,
$\frac{cE}{r^2}$ = 0.075 AND 0.15 FOR UPPER LIMIT CURVES,
$\frac{cE}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES.
FLAT ENDS.
$\frac{cE}{r^2}$ = 0.6.
$\frac{cE}{r^2}$ = 0.075
$\frac{cE}{r^2}$ = 0.15
FIXED ENDS.
FIXED ENDS.
FIXED ENDS.
RATIO $\frac{l}{r}$
RECTANGULAR SECTION RATIO $\frac{l}{d}$
f_d IN POUNDS PER SQUARE INCH.
6,000-F_c

Fig. 48.

single sticks only are shown on this diagram. The built posts tested have been omitted. Three second tests of the same sticks are not plotted. Sixteen tests by Professor Lanza made at Watertown for the Boston Manufacturers' Mutual Fire Insurance Company. The flat-ended tests only are plotted.* Fourteen tests by Kirkaldy, London, England, on pitch-pine blocks with flat ends.†

Total number of tests on diagram = 99.

The inclusion of Kirkaldy's tests of pitch pine in the same diagram as the Watertown tests and Professor Lanza's tests of yellow pine is justified by a comparison of the results obtained by Kirkaldy and Lanza on short blocks. The comparison is shown in Table No. 2.

TABLE No. 2.

KIRKALDY.		LANZA.	
Round, 9.78 ins. diameter, 50 ins. long.		Round, 7.70 to 10.46 ins. diameter; Rectangular, 8.98 × 9.02 ins. And 10.20 × 10.07 ins.	Length, 23.8 to 24.33 ins
COMPRESSIVE STRENGTH, in pounds per square inch.			
Minimum	3 586	Minimum	3 604
Maximum	5 438	Maximum	5 452
Mean of 5 tests	4 689	Mean of 8 tests	4 658
MODULUS OF ELASTICITY.			
Average up to load of 2 400 lbs. per square inch	2 000 000 lbs.	Mean of 4 tests	1 911 385 lbs
(*Minutes of Proceedings*, Institution of Civil Engineers, Vol. liii, p. 158.)		(Lanza's "Applied Mechanics," p. 351.)	

Fig. 44.—Sixty-six tests of rectangular flat-ended white pine columns, made at Watertown Arsenal.‡ The single sticks only are plotted on the diagram. The built posts tested have been omitted.

Fig. 45.—This diagram represents:

Thirty-six tests of square flat-ended seasoned French oak columns, by Lamandé;§ and thirteen tests of square flat-ended Dantzic oak columns, by Hodgkinson,‖ a total of forty-nine tests.

* Lanza's "Applied Mechanics," pp. 651 and 665, 5th edition.

† From Kirkaldy's Reports (Kirkaldy's Life) and *Minutes of Proceedings*, Institution of Civil Engineers, Vol. liii, p. 158.

‡ Exec. Doc. 12, 47th Congress, 1st Session. Plotted from the figures given in Lanza's "Applied Mechanics," pages 659 to 662, 5th edition.

§ Plotted from the figures given in Hurst's "Tredgold's Carpentry," page 84, 6th Edition, 1888.

‖ *Philosophical Transactions*, Royal Society, London, 1840.

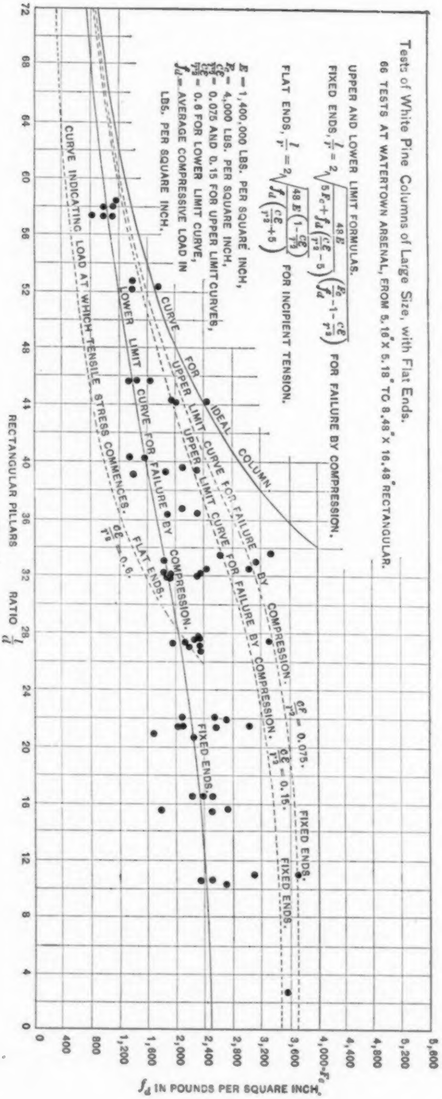

Tests of White Pine Columns of Large Size, with Flat Ends.
66 TESTS AT WATERTOWN ARSENAL, FROM 5.16" X 5.18" TO 8.48" X 16.48" RECTANGULAR.
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS, $\frac{l}{r} = 2\sqrt{\frac{48E}{5F_c + f_d\left(\frac{cE}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{cE}{r^2}\right)}$ FOR FAILURE BY COMPRESSION.
FLAT ENDS, $\frac{l}{r} = 2\sqrt{\frac{48E\left(1 - \frac{cE}{r^2}\right)}{f_d\left(\frac{cE}{r^2} + 5\right)}}$ FOR INCIPIENT TENSION.
E = 1,400,000 LBS. PER SQUARE INCH,
F_c = 4,000 LBS. PER SQUARE INCH,
$\frac{cE}{r^2}$ = 0.075 AND 0.15 FOR UPPER LIMIT CURVES,
$\frac{cE}{r^2}$ = 0.6 FOR LOWER LIMIT CURVE,
f_d = AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. $\frac{cE}{r^2}$ = 0.075. FIXED ENDS.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION. $\frac{cE}{r^2}$ = 0.15. FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES. $\frac{cE}{r^2}$ = 0.6. FLAT ENDS.
RECTANGULAR PILLARS RATIO $\frac{l}{d}$
f_d IN POUNDS PER SQUARE INCH
4,000 = F_c

FIG. 44.

In attempting to fit the curves to these three diagrams of tests of timber columns, the writer has not lost sight of the great variation known to exist in the character of specimens of timber, even from the same tree, due to a number of causes and conditions to which it is not necessary to allude. The tests plotted, however, are in each case the most important series of which the writer is aware, and they comprise the most reliable data available for the design of compression members of timber.

There have been other tests made on large-sized timbers, but they are too few in number to be of use in any endeavor to determine the influence of length on column strength. Notwithstanding the difficulties attaching to the question, it will be seen from the diagrams that the tests do, in a considerable measure, exhibit the general characteristics accompanying the numerous tests on the more uniform materials previously dealt with.

These diagrams complete the comparison attempted to be made between the results of tests of ultimate strength and the writer's rational formulas.

It may be remarked that reference has been made in the diagrams to no less than 1 620 results, representing 1 789 tests, of ultimate strength of columns of cast iron, wrought iron, steel, and timber, exclusive of the references to tests of deflection.

Application of the Formulas in Practice.

The opinion has already been expressed in the paper that the design and proportioning of columns to carry given loads should be based on the maximum fiber stresses allowable, and on the stiffness of the material, while making reference to experimental results in order to ensure having sufficient margin against ultimate failure.

In order to put this opinion into practical operation, it is necessary to adopt suitable values for the three fundamental factors in the formulas, viz., ε, F_c, and E.

It will not be necessary, in the case of "centrally" loaded columns, in practice to deal with the value of the allowable tensile stress, since tensile stress will not be developed under ordinary working loads.

The value of ε is that concerning which our knowledge must be drawn entirely from the comparisons just made with experimental re-

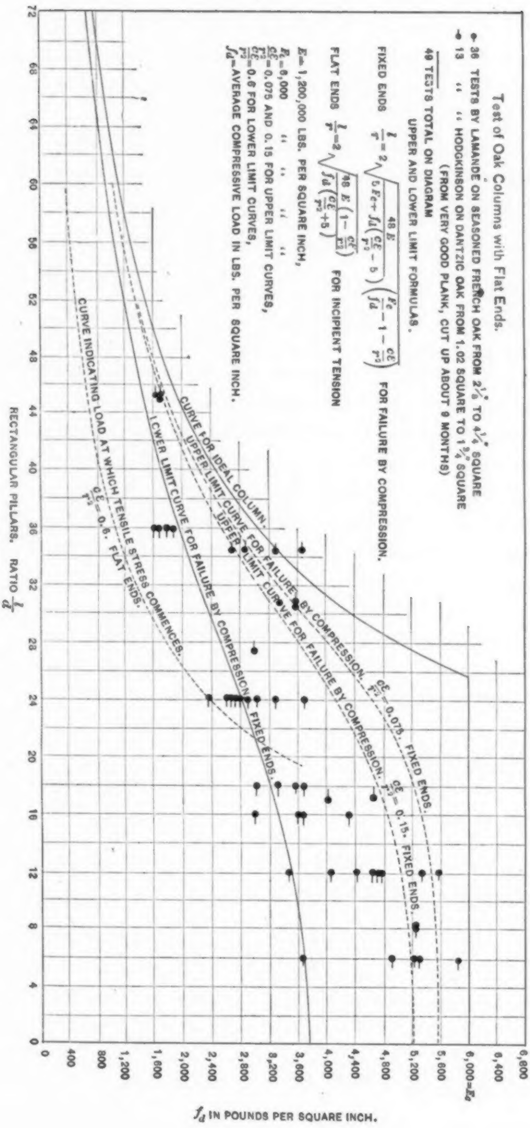

Test of Oak Columns with Flat Ends.
36 TESTS BY LAMANDÉ ON SEASONED FRENCH OAK FROM 2⅛ TO 4¼ SQUARE
13 " " HODGKINSON ON DANTZIC OAK FROM 1.02 SQUARE TO 1¾ SQUARE
(FROM VERY GOOD PLANK, CUT UP ABOUT 9 MONTHS)
49 TESTS TOTAL ON DIAGRAM
UPPER AND LOWER LIMIT FORMULAS.
FIXED ENDS
FOR FAILURE BY COMPRESSION.
FLAT ENDS
FOR INCIPIENT TENSION
E= 1,200,000 LBS. PER SQUARE INCH,
Ec=6,000 " " " "
=0.075 AND 0.15 FOR UPPER LIMIT CURVES,
=0.6 FOR LOWER LIMIT CURVES,
fd=AVERAGE COMPRESSIVE LOAD IN LBS. PER SQUARE INCH.
CURVE FOR IDEAL COLUMN.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
= 0.075. FIXED ENDS.
UPPER LIMIT CURVE FOR FAILURE BY COMPRESSION.
= 0.15. FIXED ENDS.
LOWER LIMIT CURVE FOR FAILURE BY COMPRESSION. FIXED ENDS.
CURVE INDICATING LOAD AT WHICH TENSILE STRESS COMMENCES.
= 0.6. FLAT ENDS.
RECTANGULAR PILLARS. RATIO l/d
fd IN POUNDS PER SQUARE INCH.
0 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72
0 400 800 1,200 1,600 2,000 2,400 2,800 3,200 3,600 4,000 4,400 4,800 5,200 5,600 6,000=Ec 6,400 6,800

FIG. 45.

sults, and, although the value of $\frac{c}{r^2}$ can be at once calculated from the section of column proposed to be used, yet it appears desirable in the case of "centrally" loaded columns that one constant value should be adopted for the quantity $\frac{c\,\varepsilon}{r^2}$; and judging from experimental evidence, the value to be assigned to it should not be taken at less than 0.6, in the present state of our knowledge.

In the case of a solid cylindrical column, this value of $\frac{c\,\varepsilon}{r^2}$ corresponds to an equivalent eccentricity of only 0.3 of the radius of gyration.

The values to be given to F_c for different materials may properly be approximately those in common use for maximum working fiber stresses in solid beams under transverse stress.

With regard to the value to be given to E, it must be noted that in every column it is necessary, not only to provide that the maximum fiber stress shall be confined within proper limits, but also that the stability of the column, as governed by the stiffness of the material of which it is made, shall also have a proper and sufficient margin of safety under the loads to be imposed.

The necessity for this is at once seen by a reference to the calculations of deflection and resulting maximum fiber stresses of a long column, given on page 351, and in connection with which it was shown that the column would be perfectly safe as regards the intensity of maximum fiber stress under a load amounting to over 90% of the ultimate supporting power.

Now, on reference to Formula (1), the general expression for the deflection of a column:

$$\Delta = \frac{P\,l^2\,e}{8\,E\,I - 2\,P\,l^2\,y\,x} \qquad (1)$$

or its practical modification

$$\Delta = \frac{P\,l^2\,e}{8\,E\,I - \frac{5}{6}\,P\,l^2} \qquad (2)$$

it will be seen that the deflection of any given column under any given load increases directly as the eccentricity e, and further, that if we had a material to deal with, such that we need give no thought to the maximum stress developed, we would have the curious result that the ulti-

mate load would be precisely the same whatever the amount of eccentricity might be.

The practical conclusion to be drawn from the foregoing is that the eccentricity of loading has nothing to do with the stability or instability of the column, if the condition of instability be defined as that which occurs when the deflection under load becomes infinite.

From this we see that in designing a column to carry a given load we have two totally independent modes of failure to guard against:

(1) Against failure by excessive intensity of fiber stress.

(2) Against failure by instability.

In the first we are dependent on strength, and in the second on stiffness.

We will proceed, therefore, on perfectly correct rational lines if we proportion our "centrally" loaded columns in practice to meet these two conditions, which are satisfied by using the two formulas:

For columns with both ends round
$$\frac{l}{r} = \sqrt{\frac{48\,E}{5\,F_c + f_d\left(\frac{c\,\varepsilon}{r^2} - 5\right)}\left(\frac{F_c}{f_d} - 1 - \frac{c\,\varepsilon}{r^2}\right)} \quad ..(7)$$

where F_c is the maximum allowable working intensity of fiber stress,

$$P = K\frac{9.6\,E\,I}{l^2} \text{ for stability} \quad ....(2)$$

where K is a suitable fractional coefficient of safety,

or, since $f_d = \frac{P}{a}$, $f_d = K\frac{9.6\,E r^2}{l^2} = \frac{9.6\,K E}{\frac{l^2}{r^2}}$.

Applying these formulas to the material which is in most general use in constructional work at the present day, *i. e.*, mild steel, and adopting as suitable values for the various factors,

$E = 30\,000\,000$ lbs.;

$F_c = 24\,000$ lbs. per square inch, working dead stress;

$\frac{c\,\varepsilon}{r^2} = 0.6$ for columns in which $\frac{c}{r} = 2$, and $\frac{\varepsilon}{r} = 0.3$;

K = coefficient of safety against instability, say $\frac{1}{3}$;

we will have the working loads per square inch on solid, cylindrical, mild-steel, round-ended columns indicated by the curves in Fig. 46, in which, for every ratio $\frac{l}{r}$, the conditions are satisfied that the maximum fiber stress shall not be more than 24 000 lbs. per square inch,

and the working load shall not be more than one-third of that which will cause instability.

It will be noticed that, at about ratio $\frac{l}{r} = 88$, the two curves intersect, and at this point the factor against ultimate failure, as compared with experimental results, is somewhat less than is given either for very short or very long columns, and however rational the basis upon which the results depend, the writer thinks that, like himself, other engineers would hesitate to adopt heavier loads, in relation to ultimate

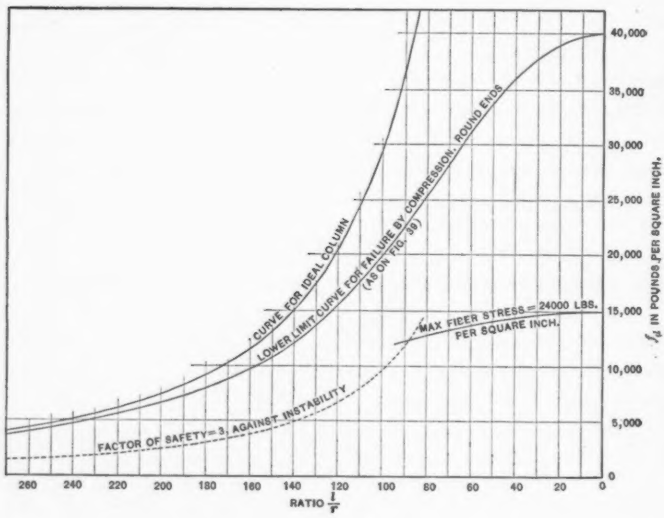


FIG. 46.

strength for a column 89 or 90 radii long, than would be considered safe for columns only one-quarter of the length, or four times the length.

Again, it may reasonably be objected that if we require the factor of safety against failure by instability of a long column to be at least 3 (assuming this as a fair value), it should follow that we would not have a much smaller value against ultimate strength as shown by experiment, in the case of columns of ratio $\frac{l}{r} = 80$ to 90.

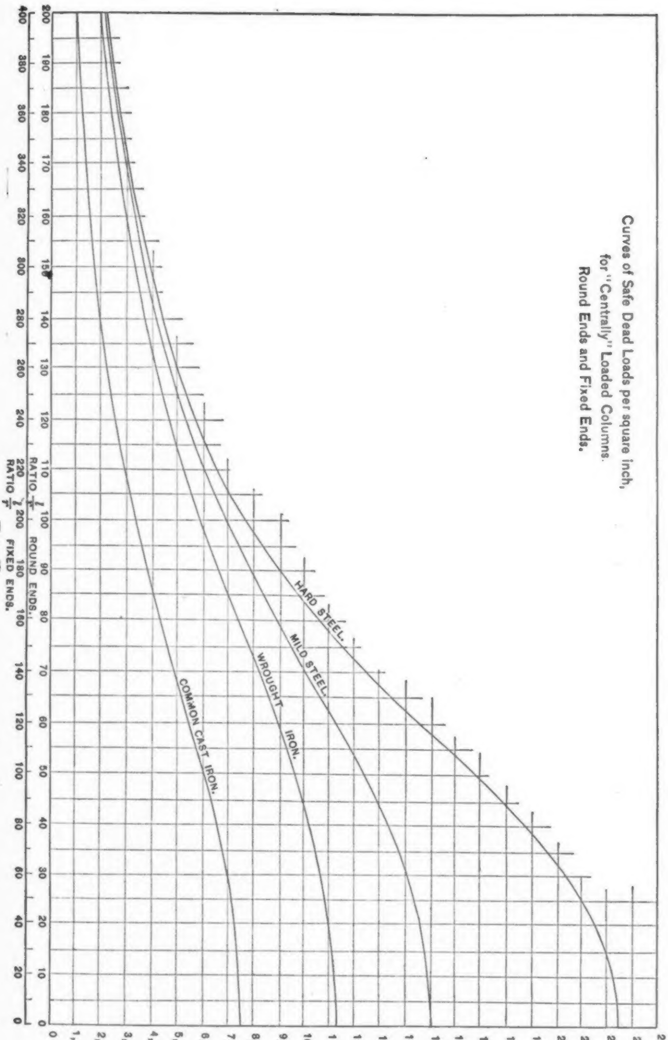

Curves of Safe Dead Loads per square inch,
for "Centrally" Loaded Columns.
Round Ends and Fixed Ends.
HARD STEEL.
MILD STEEL.
WROUGHT IRON.
COMMON CAST IRON.
RATIO $\frac{l}{r}$ ROUND ENDS.
RATIO $\frac{l}{r}$ FIXED ENDS.
f_d IN POUNDS PER SQUARE INCH.
0
1,000
2,000
3,000
4,000
5,000
6,000
7,000
8,000
9,000
10,000
11,000
12,000
13,000
14,000
15,000
16,000
17,000
18,000
19,000
20,000
21,000
22,000
23,000
24,000

Fig. 47.

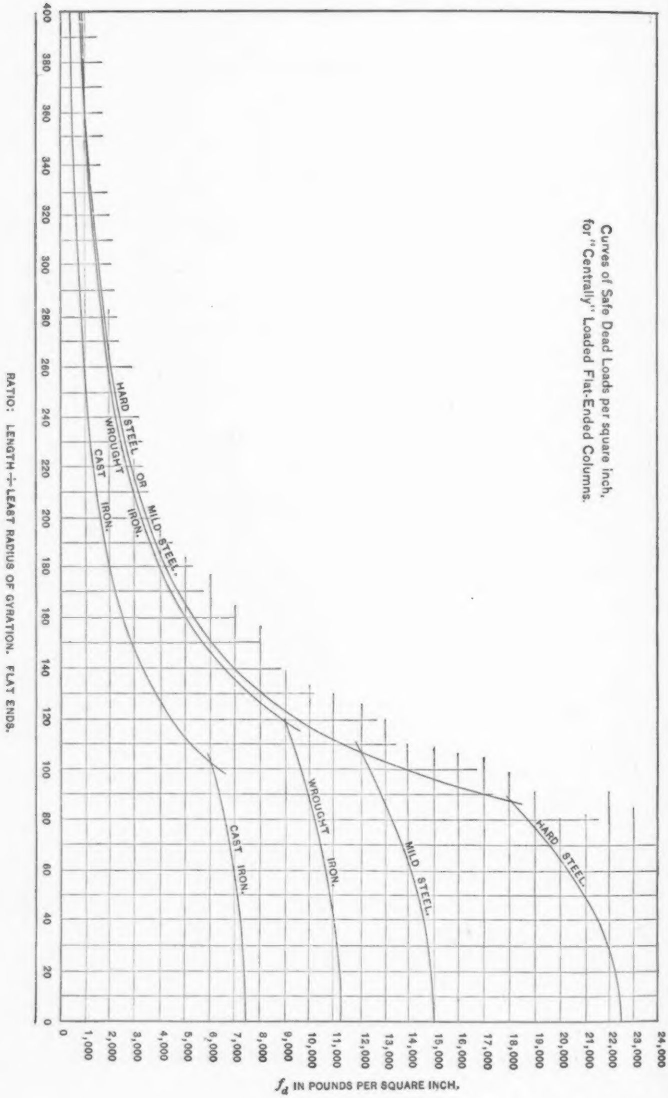

Curves of Safe Dead Loads per square inch,
for "Centrally" Loaded Flat-Ended Columns.
HARD STEEL OR MILD STEEL.
WROUGHT IRON.
CAST IRON.
CAST IRON.
WROUGHT IRON.
MILD STEEL.
HARD STEEL.
RATIO: LENGTH ÷ LEAST RADIUS OF GYRATION. FLAT ENDS.
f_d IN POUNDS PER SQUARE INCH.
0
20
40
60
80
100
120
140
160
180
200
220
240
260
280
300
320
340
360
380
400
0
1,000
2,000
3,000
4,000
5,000
6,000
7,000
8,000
9,000
10,000
11,000
12,000
13,000
14,000
15,000
16,000
17,000
18,000
19,000
20,000
21,000
22,000
23,000
24,000

Fig. 48.

TABLE No. 3.—WORKING FORMULAS FOR COLUMNS UNDER "CENTRAL LOADING."

(1)	(2)	(3)	(4)	(5)
MATERIAL OF COLUMN.	$F_c =$ Maximum compressive fiber stress. Pounds per square inch dead load.	$KE = \frac{\text{Mod. Elas.}}{3}$ Pounds per square inch.	Value of $\frac{c\,\epsilon}{r^2}$	Working formulas for round-ended or pivot-ended columns. $\frac{l}{r} = \sqrt{\frac{48\,K\,E}{5\,F_c + f_d\left(\frac{c\,\epsilon}{r^2} - 5\right)}\left[\frac{F_c}{f_d} - 1 - \frac{c\,\epsilon}{r^2}\right]}$
Common cast iron........................	12 000 = 1.2 units.	$\frac{14\ 000\ 000}{3}$ = 466.6 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{224}{6 - 4.4 f_d}\left(\frac{1.2}{f_d} - 1.6\right)}$
Wrought iron.............................. Tensile strength, 45 000 to 50 000 lbs. per square inch.	18 000 = 1.8 units.	$\frac{28\ 000\ 000}{3}$ = 933.3 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{448}{9 - 4.4 f_d}\left(\frac{1.8}{f_d} - 1.6\right)}$
Mild steel.................................. Tensile strength, 60 000 to 70 000 lbs. per square inch.	24 000 = 2.4 units.	$\frac{30\ 000\ 000}{3}$ = 1 000 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{480}{12 - 4.4 f_d}\left(\frac{2.4}{f_d} - 1.6\right)}$
Hard steel.................................. Tensile strength, about 100 000 lbs. per square inch.	36 000 = 3.6 units.	$\frac{30\ 000\ 000}{3}$ = 1 000 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{480}{18 - 4.4 f_d}\left(\frac{3.6}{f_d} - 1.6\right)}$
Timber (Unit for F_c, f_d and E = 1 000 lbs.)				
Yellow pine or pitch pine...............	2 000 = 2 units.	$\frac{2\ 200\ 000}{3}$ = 733.3 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{352}{10 - 4.4 f_d}\left(\frac{2}{f_d} - 1.6\right)}$
White pine.................................	1 300 = 1.3 units.	$\frac{1\ 400\ 000}{3}$ = 466.6 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{224}{6.5 - 4.4 f_d}\left(\frac{1.3}{f_d} - 1.6\right)}$
French oak or Dantzic oak.............	2 000 = 2 units	$\frac{1\ 200\ 000}{3}$ = 400 units.	0.6	$\frac{l}{r} = 10\sqrt{\frac{192}{10 - 4.4 f_d}\left(\frac{2}{f_d} - 1.6\right)}$

In Column (5) a unit of 10 000 lbs. is used for F_c, f_d and E for metals, and 1 000 lbs. for timber.

This objection may be met very easily and simply by using only Formula (7), with the modification of applying the factor of safety against instability to the value of E in that formula, which then becomes:

$$\frac{l}{r}=\sqrt{\frac{48\,K E}{5\,F_c+f_d\left(\frac{c\,\varepsilon}{r^2}-5\right)}\left[\frac{F_c}{f_d}-1-\frac{c\,\varepsilon}{r^2}\right]},$$

$$\text{where } K=\frac{1}{\text{factor of safety.}}$$

It will be seen that the actual result of this formula is to give the ratio $\frac{l}{r}$ corresponding to a given average load per square inch for material in which the maximum compressive fiber stress is limited to F_c, and for which the value of the modulus of elasticity is taken to be $K E$, instead of the actual value E pertaining to the material.

The use of the coefficient K in this manner has an inappreciable influence on the results of the formulas when applied to short columns, while its effect gradually increases with the length, ultimately affording a factor of safety of 3, against failure by instability, in the case of exceedingly long columns, and the factor of safety against ultimate strength is fairly even between these extremes.

Table No. 3 gives the resulting formulas reduced by the insertion of suitable values of $\frac{c\,\varepsilon}{r^2}$, F_c and $K E$.

For fixed-ended columns the values of $\frac{l}{r}$ obtained by the formulas in Table No. 3 are to be doubled, or in other words, for a given strength, a fixed-ended column is twice as long as a round-ended or pivot-ended one.

For flat-ended columns, the relation between the strength and the ratio $\frac{l}{r}$ is the same as for fixed-ended columns, up to the point when incipient tension is the controlling factor, and it is then necessary to use the modified formulas as previously described for that condition, except that, as we are now dealing with safe working loads, the coefficient of safety K must be inserted in Formula (9), which then becomes

$$\frac{l}{r}=2\sqrt{\frac{48\;K\;E\left(1-\frac{c\,\varepsilon}{r^2}\right)}{f_d\left(\frac{c\,\varepsilon}{r^2}+5\right)}}$$

or inserting

$$K = \tfrac{1}{3} \text{ and } \frac{c\,\varepsilon}{r^2} = 0.6,$$

$$\frac{l}{r} = 2.138\sqrt{\frac{E}{f_d}}$$

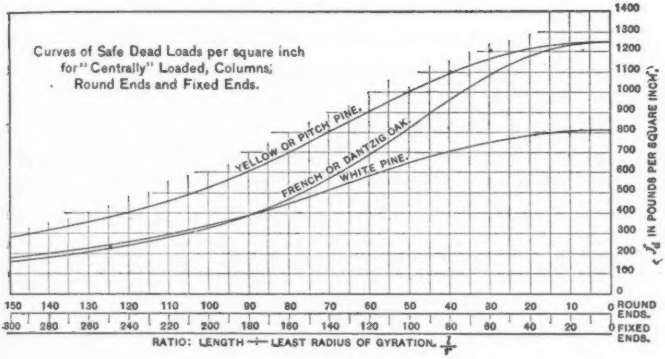


FIG. 49.

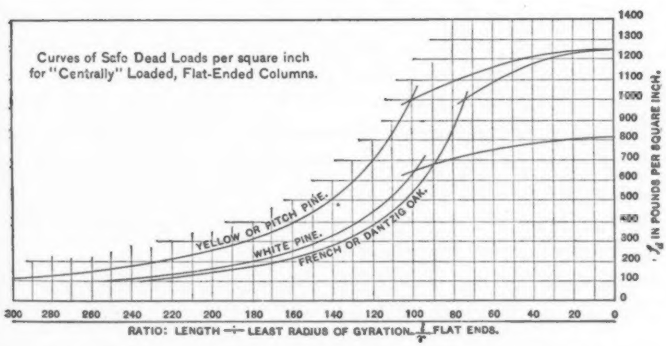


FIG. 50.

The results of the various reduced formulas for working dead loads in Table No. 3 for round-ended columns, and of the modifications above stated for fixed and flat-ended columns, are shown on various diagrams accompanying the paper, and are also shown in Figs. 47, 48, 49 and 50, drawn to a larger vertical scale for convenience in use.

With regard to the curves for working dead-loads on fixed-ended and flat-ended columns, it must be remembered that in a laboratory or test-room experiment on a fixed or flat-ended column, the conditions of end fixing are totally different from those met in ordinary practice. The rigidity of the testing machine bearing faces, as compared with the column under test, is far greater than is afforded by any end bearing or connection in practice, and for this reason it is well not to count, in actual work, upon such high strengths as are indicated by experiment.

The only modification necessary in dealing with live loads or moving loads instead of the dead loads for which the stresses adopted in Table No. 3 are suitable, consists obviously of a reduction in the value of F_c or F_t, and an increase in the value of K, although in the writer's practice he adopts the method of increasing the moving load by a suitable percentage, dependent on the character of the load, and then treats the result as a dead-load equivalent.

Columns Under Intentionally Eccentric Loads.

The experimental data, to which reference can be made in regard to columns on which the load is imposed with large eccentricity as compared with the size of column, are exceedingly few, and as far as the writer is aware, are limited to the tests by Tetmajer (Figs. 24 and 25); but as the theoretical principles upon which the formulas are based are those in common use and acceptance in all cases involving transverse bending simply combined with the influence of direct loading, there can be no more hesitation in applying them to columns than there is in any case of simple bending.

In making use of the formulas in practice, it is, however, not only sufficient to take into account the actual measured eccentricity and the value of $\frac{c}{r^2}$ as fixed by the section, but allowance should be made for the "accidental" value of $\frac{c\,\varepsilon}{r^2}$, as dealt with in the case of presumably central loading.

This being the case, it will be necessary to add the value of $\frac{c\,\varepsilon}{r^2} =$ 0.6, as determined for centrally-loaded columns, to the measured

value of $\frac{c\,e}{r^2}$, as obtained from the intended eccentricity e, and the dimensions and form of the sections which fix the value of $\frac{c}{r^2}$.

In this way a perfectly rational recognition is given to the form of section, in precisely the same manner as in the case of beams under simple flexure, and the formulas for the relation between the ratio $\frac{l}{r}$ and the average load per square inch for a free-ended column carrying an eccentric load thus become, for safe working loads,

$$\frac{l}{r}=\sqrt{\frac{48\,K\,E}{5\,F_c+f_a\left\{\left(\frac{c\,e}{r^2}+0.6\right)-5\right\}}\left[\frac{F_c}{f_a}-1-\left(\frac{c\,e}{r^2}+0.6\right)\right]}$$

where F_c is the maximum allowable working fiber stress in compression, or

$$\frac{l}{r}=\sqrt{\frac{48\,K\,E}{5\,F_t-f_a\left\{\left(\frac{c\,e}{r^2}+0.6\right)+5\right\}}\left[\frac{F_t}{f_a}-1+\left(\frac{c\,e}{r^2}+0.6\right)\right]}$$

where F_t is the maximum allowable tensile fiber stress (and being tensile it must be accorded the minus sign independently of the fixed signs in the formula, as has been pointed out previously).

These formulas again reduce to very simple expressions when the values of $K\,E$, F_c or F_t and $\frac{c\,e}{r^2}$ are inserted, as will be seen by the following application to mild steel, using the same values for the various quantities as adopted for "centrally" loaded columns.

Since the maximum compressive stress actually developed in a column of symmetrical section always exceeds the maximum tensile stress, it will not be necessary, in the case of mild steel, to use the formula containing the value of maximum allowable tensile stress F_t, as this may be taken, for this material under working loads, as having the same value accorded to F_c, the allowable compressive fiber stress.

In the case of cast iron it will, of course, be necessary to use both expressions, and to adopt whichever gives the lower value to the ratio $\frac{l}{r}$ for any given load f_a.

The formula, then, for the following values of $\left(\frac{c\,e}{r^2}+0.6\right)$ inserted, will be for mild-steel columns of symmetrical section, carrying dead loads, using 10 000 lbs. as a unit for E, F_c and f_d:

$$\text{When } \left(\frac{c\,e}{r^2}+0.6\right)=2,\ \frac{l}{r}=10\sqrt{\frac{480}{12-3f_d}\left(\frac{2.4}{f_d}-3\right)}$$

$$\text{“ “ } =4,\ \frac{l}{r}=10\sqrt{\frac{480}{12-f_d}\left(\frac{2.4}{f_d}-5\right)}$$

$$\text{“ “ } =6,\ \frac{l}{r}=10\sqrt{\frac{480}{12+f_d}\left(\frac{2.4}{f_d}-7\right)}$$

$$\text{“ “ } =8,\ \frac{l}{r}=10\sqrt{\frac{480}{12+3f_d}\left(\frac{2.4}{f_d}-9\right)}$$

The working out of these values of the formula is shown by the curves on Fig. 51.

A reference to the remarks previously made in connection with Fig. 24 will show that in dealing with columns of unsymmetrical section it may be necessary to consider the question of tensile stress, and to use the modification of the formula suitable to such conditions.

Columns Subjected to Side Loads.

The effect of a side load in addition to the end load on a column is easily traced by the aid of the principles and formulas in the foregoing pages. To illustrate this it will be convenient to take a direct example.

Assume a mild-steel column with pivot ends, and having the following particulars:

$l = 100$ ins. = length;
$I = 10$ ins.^{4};
$r = 1$ in.;
$c = 2.5$ ins., from axis to extreme fibers;
Sectional area = 10 sq. ins.;
$E = 30\,000\,000$ lbs.;

and assume the side load to be 4 000 lbs., distributed along the column's length.

This will produce a bending moment at the center,

$$M=\frac{4\,000 \text{ lbs.} \times 100 \text{ ins.}}{8}=50\,000 \text{ inch-pounds,}$$

and the resulting maximum fiber stress will be

$$f=\frac{Mc}{I}=\frac{50\,000\times 2.5}{10}=\pm\ 12\,500 \text{ lbs. per square inch.}$$

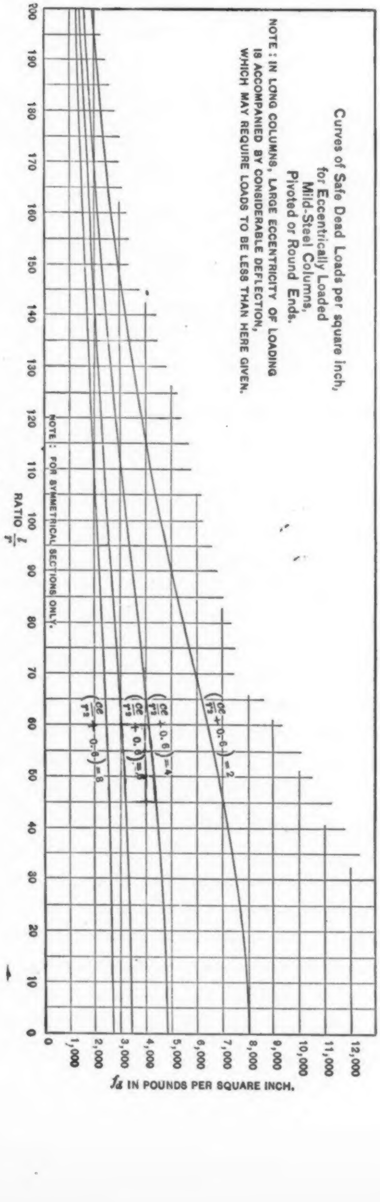

Curves of Safe Dead Loads per square inch,
for Eccentrically Loaded
Mild-Steel Columns,
Pivoted or Round Ends.
NOTE: IN LONG COLUMNS, LARGE ECCENTRICITY OF LOADING
IS ACCOMPANIED BY CONSIDERABLE DEFLECTION,
WHICH MAY REQUIRE LOADS TO BE LESS THAN HERE GIVEN.
NOTE: FOR SYMMETRICAL SECTIONS ONLY.
$\left(\frac{ce}{r^2}+0.6\right)=2$
$\left(\frac{ce}{r^2}+0.6\right)=4$
$\left(\frac{ce}{r^2}+0.6\right)=6$
$\left(\frac{ce}{r^2}+0.6\right)=8$
RATIO $\frac{l}{r}$
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200
0 1,000 2,000 3,000 4,000 5,000 6,000 7,000 8,000 9,000 10,000 11,000 12,000
f_s IN POUNDS PER SQUARE INCH.

Fig. 51.

The side-load bending moment diagram will have a parabolic outline, as shown in Fig. 52, the central depth being 50 000 inch-pounds to any convenient scale.

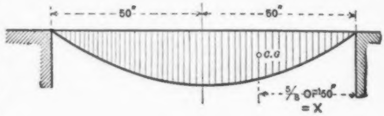


FIG. 52.

Treating each half of the column (or beam) as a cantilever, with respect to its central section, the deflection at the center will be

$$v = \frac{A\,X}{E\,I},$$

where $A = 50\,000 \times 50$ ins. $\times \frac{2}{3}$, and $X = \frac{5}{8}$ of 50 ins., and therefore,

$$v = \frac{(50\,000 \times 50 \times \frac{2}{3}) \times (\frac{5}{8} \text{ of } 50)}{30\,000\,000 \times 10} = 0.1736 \text{ in.}$$

In order to determine the deflection of the column under end loads it is now only necessary to treat it in precisely the same way as already indicated for a column having an initial curvature, with the value of central versed sine, $v = 0.1736$ in., keeping in view, when considering its strength as against end loads, that we have already absorbed 12 500 lbs. of the allowable maximum fiber stress.

In the present case it is not difficult to determine the precise theoretic form of the curve assumed by the bent column under its side load alone, this curve depending on the disposition of the load.

In practical work, however, it is not necessary to go to this degree of accuracy, and we may assume the curve to be a parabola, without involving any serious error, and the justification is found in the fact that, had we assumed the side load to be concentrated at the center, the assumption of a parabola instead of the precise curve taken by the bent column would only involve an excess, as regards its influence on the deflection, of a little over 4 per cent. As compared with the curve produced by a distributed load, the parabola would have a much less difference than even that named.

In order to make the comparison between the column with side load, and one of ordinary character with presumably central loading, it will be necessary to treat both on the same lines as far as possible, and in calculating the safe dead load on the centrally-loaded column, we have limited the maximum fiber stress to 24 000 lbs. per square inch, and have at the same time used a value of

$$K\,E = \frac{30\,000\,000}{3} = 10\,000\,000 \text{ lbs.},$$

and also a value of $\frac{c\,\varepsilon}{r^2} = 0.6$.

These same values must be used also in the side-loaded columns as regards the influence of the end loads at least.

The value of $\frac{c\,\varepsilon}{r^2} = 0.6$, in the case of this column with a radius of gyration $= 1$ in., corresponds to an equivalent eccentricity of 0.24 in., and we thus have a column with an initial curvature measured by a versed sine of $v = 0.174$ in. and an eccentricity of loading $\varepsilon = 0.24$ in., and making use of Formula (3) (with K inserted) to determine the deflection caused by different values of the end load, P, we have

$$\varDelta = \frac{P\,l^2\,(\varepsilon + 2\,y_1\,x_1\,v)}{8\,(K\,E)\,I - 2\,P\,l^2\,y\,x}.$$

We have already assumed the curve of the bent column, from both side and end loads, to be a parabola, and the values of $2\,y_1\,x_1$ and $2\,y\,x$ will each be $\left(2 \times \frac{2}{3} \times \frac{5}{8}\right) = \frac{5}{6}$, and, inserting these and the values of the other factors in the formula, we have:

$$\varDelta = \frac{P \times 10\,000\left(0.24 + \frac{5}{6} \times 0.174\right)}{(8 \times 10\,000\,000 \times 10) - \left(\frac{5}{6} P \times 10\,000\right)} = \frac{0.462\,P}{96\,000 - P} \text{ ins.},$$

for the elastic deflection of this column under any end load P.

TABLE No. 4.

P, in pounds.	$\Delta = \frac{0.462\,P}{96\,000 - P}$	Bending moment from end loads. $M = P\,(0.414 + \Delta)$ inch-pounds.	Maximum fiber stress from end-load bending moments. Pounds per square inch.	Direct stress. $f_a = \frac{P}{a}$. Pounds per square inch.	Maximum fiber stress from side-load bending moments. Pounds per square inch.	Total maximum compressive fiber stress. Pounds per square inch.
30 000	0.21	18 720	4 680	3 000	12 500	20 180
32 000	0.231	20 640	5 160	3 200	12 500	20 860
34 000	0.253	22 680	5 670	3 400	12 500	21 570
36 000	0.277	24 880	6 220	3 600	12 500	22 320
38 000	0.303	27 240	6 810	3 800	12 500	23 110
40 000	0.33	29 760	7 440	4 000	12 500	23 940

The bending moment at the center of the column, from end loads alone, will be $M = P\,(\varepsilon + v + \varDelta) = P\,(0.24 + 0.174 + \varDelta)$

$$= P\,(0.414 + \varDelta) \text{ ins.};$$

and now calculating the deflections, bending moments, and resulting maximum fiber stresses caused by the end loads, and combining them with the maximum fiber stress of 12 500 lbs. caused by the end loads alone, we have the results given in Table No. 4.

From these results we see that with an end load of 4 000 lbs. per square inch, the maximum fiber stress reaches the limiting value of 24 000 lbs. per square inch, and, on referring to Fig. 47, we find that with the presumably centrally-loaded column, under the same basis of calculation, the safe load f_a would be (for the ratio $\frac{l}{r} = 100$) 7 000 lbs. per square inch, or in other words, the side load has reduced the safe end load by nearly 43 per cent.

The foregoing calculations will serve to illustrate the proper mode of treating the conditions met with in side-loaded columns, although they have been applied to a column of such proportions as would rarely be trusted to carry any appreciable amount of side load other than its own weight.

The values of I, r and c, and the sectional area assumed, are roughly those of a rolled girder section 12 × 5 ins. × 33 lbs. per foot.

The formulas were deduced by the writer in 1896 and 1897, for use in his own practice, and he hopes that the results given in this paper may be found to be fairly in accordance with the leading principles set forth in the opening sentences as being those with which a column theory and its resulting formulas should comply, at least from the point of view of the practicing engineer rather than that of the pure mathematician.

The question of the proper construction of columns so as to ensure the full development of the strength which may rightfully be expected from any given section, is one which has not been dealt with herein, although it is certainly of the greatest importance, in view of the inferior types sometimes adopted in present-day practice.

Of these inferior types the writer would especially draw attention to one which appears to be in more common use, and that is, the column with batten-plate connections between the main members forming the column, as a substitute for a properly constructed web,

and a reference to the illustrations of the failure of a bridge in Servia,* will give ample evidence of the inefficiency of the type. Its inferiority cannot be questioned in the case of columns carrying eccentric loads, or having to resist shearing stresses from side wind-loads as in some of the lofty buildings constructed in America.

The writer is conscious that in these pages there will be found (and necessarily so) many points of resemblance and identity, both in views and in their expression, with the contributions of others, and he desires to acknowledge his indebtedness to the numerous writers from whose papers he has drawn information and assistance.

While, from a strictly mathematical point of view, objections may possibly be raised to the reasoning and results in this paper, yet if a close comparison be made with more highly analytical investigations, it is believed that the differences from the latter will be found to be of little importance in any practical sense.

The writer was impelled to attempt the deduction of new formulas by having to design a number of columns to carry heavy loads with exceptionally large eccentricity, for which condition he could find no theory or formulas sufficiently simple and easy of application under varied circumstances.

The concentration of such a large mass of scattered experimental evidence will, it is hoped, be of sufficient value to justify the labor involved in its collection, examination and arrangement in diagram form.

* *Engineering*, February 3d, 1893.

DISCUSSION.

Mr. Jonson. ERNST F. JONSON, Assoc. M. Am. Soc. C. E. (by letter).—The author states that the curve of flexure of a column of uniform cross-section and elasticity is a curve of sines only when the eccentricity of loading is infinitely small.

The writer, therefore, would call attention to the fact that the nature of the curve is in nowise changed by increasing the eccentricity. Considering the line in which the external forces act, as the axis of abscissas, the bending moment is still proportional to the ordinate. Hence, no matter how great the eccentricity may be, the curve is a curve of sines, the equation of which is

$$y = D \cos. \frac{x}{r}\sqrt{\frac{w}{E}} \qquad \text{I}$$

where x is abscissa, y ordinate, D maximum ordinate, r radius of gyration, w load per unit of cross-section area, and E modulus of elasticity.

For maximum w

$$D = \frac{r^2\, s}{a\, w} \qquad \text{II}$$

where a is distance to extreme fiber and $s = p - w$ when $p - w \overline{\overline{<}}\ t + w$, or $= t + w$ when $p - w > t + w$, p being the proportional limit in compression and t the proportional limit in tension.

Hence,

$$y = \frac{r^2\, s}{a\, w} \cos. \frac{x}{r}\sqrt{\frac{w}{E}} \qquad \text{III}$$

If y is given, the value of $e = c + b$, c being the equivalent eccentricity, and b the intentional eccentricity, x becomes the length of the elementary column, *i. e.*, the column fixed at one end and free at the other. We have then

$$c + b = \frac{r^2\, s}{a\, w} \cos. \frac{l}{r}\sqrt{\frac{w}{E}} \qquad \text{IV}$$

or

$$l = r\sqrt{\frac{E}{w}} \cos.^{-1} \frac{a\, w}{r^2\, s}\,(c + b) \qquad \text{V}$$

Apart from its mathematical form, the foregoing formula differs from that of the author in that it makes the maximum load a function, not of the ultimate strength of the material, but of the proportional limit. When this limit has been passed at the point of greatest bending the deflection increases very rapidly, owing to the decrease of the modulus of elasticity. If the eccentricity is very small a very small addition to the load will produce an infinite deflection. If the eccentricity is larger, the extent to which the column may be strained beyond the propor-

tional limit also becomes larger. Until this point has been determined we must, however, regard the proportional limit as the determining factor in calculating the strength of columns. These facts are mentioned by the author, but he fails to draw the conclusion to which they point. Mr. Jonson.

The author argues that in designing a column it is necessary to guard against two totally independent modes of failure, viz., failure by excessive intensity of fiber stress, and failure by instability. He accordingly proposes the use of two formulas (7 and 2), or rather two forms of one formula, the second being nothing but the special form which the first assumes when the equivalent eccentricity is reduced to zero. This is clearly unnecessary. If a column is strong enough to sustain a certain load eccentrically applied, it is also strong enough when this load is applied exactly in the axis of the column.

In Fig. 46, the curve of Formula (2) falls below that of Formula (7) when $\frac{l}{r} > 88$. But this is due to the application of a factor of safety to E in the former formula. If the same factor of safety had been applied to E in both formulas, Curve 2 would always have been above Curve 7.

The formula here given was, as far as the writer knows, first applied to experimental results by A. Marston, Assoc. M. Am. Soc. C. E. Mr. Marston, the same as the author, proposes a constant value for the relation $\frac{a\,c}{r}$. This implies that a short column is more likely to be imperfect than a long one.

It seems to the writer that the amount of imperfection in a column ought to be proportional to the length, hence, since the effect of imperfection is also as the length, the equivalent eccentricity ought to be proportional to the square of the length of the column.

Now, if l^2 were introduced into the cyclometrical function of the formula, it could not be solved conveniently. We will, therefore, substitute the value $\frac{1}{w}$, thus making $c = \frac{r^2\,k}{a\,w}$, where k is the empirical constant, the coefficient of imperfection, and consequently Equation V becomes

$$l = r\sqrt{\frac{E}{w}}\cos.^{-1}\left(\frac{k}{s} + \frac{a\,b\,w}{r^2\,s}\right) \ldots\ldots\ldots\ldots\ldots \text{VI}$$

How this conclusion agrees with experience, may be seen from Table No. 5, which shows the coefficient of imperfection of some of the round-ended wrought-iron columns tested by Mr. James Christie. This table seems to show that we come much nearer the truth by assuming k to be a constant than when we make it a variable, proportional to w. A very close agreement, of course, cannot be expected.

Mr. Jonson.

TABLE No. 5.

No.	$\frac{l}{r}$	k.	Averages of k.	No.	$\frac{l}{r}$	k	Averages of k.
221	44	562		202	229	2 159	
222	44	4 430		213	239	9 164	
218	48	7 403		208	251	1 667	2 963
220	63	2 170	4 507	225	294	−1 137	
214	80	8 209		227	304	−32	
219	81	4 270		228	305	2 244	
215	103	1 905		203	306	4 627	3 205
210	113	7 640		209	310	9 111	
205	129	3 772		204	378	−76	
224	130	634		226	424	7 873	4 478
216	138	1 901		229	456	1 083	
211	145	5 542					
206	146	834	3 318				3 694
200	158	−2 166					
217	170	10 882					
201	179	2 160					
207	186	2 193					
212	195	−183					

$$k = (p - w) \cos. \frac{l}{2\,t}\sqrt{\frac{w}{E}},\ p = 38\,000,\quad E = 28\,000\,000.$$

It seems to the writer that a fractional factor of safety should be applied to each of the three empirical constants in the formula, viz., E, k and p or t. Designating this factor by K, the least of the two values $K_3\,p - w$ and $K_3\,t + w$ by s_1, we have the formulas for columns of uniform cross-section and elasticity in their final form.

One end fixed and one end free:

"Central" load;

$$\frac{l}{r} = \sqrt{\frac{K_1\,E}{w}} \cos.^{-1} \frac{k}{K_2\,s_1}$$

Eccentric load;

$$l = r\sqrt{\frac{K_1\,E}{w}} \cos.^{-1}\left(\frac{k}{K_2\,s_1} + \frac{a\,b\,w}{r^2\,s_1}\right)$$

Both ends hinged:

"Central" load;

$$\frac{l}{r} = 2\sqrt{\frac{K_1\,E}{w}} \cos.^{-1} \frac{k}{K_2\,s_1}$$

Eccentric load;

$$l = r\sqrt{\frac{K_1\,E}{w}}\left[\cos.^{-1}\frac{k}{K_2\,s_1} + \cos.^{-1}\left(\frac{k}{K_2\,s_1} + \frac{a\,b\,w}{r_2\,s_1}\right)\right]$$

One end fixed and one end hinged:

"Central" load;

$$\frac{l}{r} = 3\sqrt{\frac{K_1\,E}{w}} \cos.^{-1} \frac{k}{K_2\,s_1}$$

Eccentric load;

$$l = r\sqrt{\frac{K_1 E}{w}}\left[2\cos.^{-1}\frac{k}{K_2 s_1} + \cos.^{-1}\left(\frac{k}{K_2 s_1} + \frac{a\,b\,w}{r^2\,s_1}\right)\right]$$ Mr. Jonson.

Both ends fixed:

$$\frac{l}{r} = 4\sqrt{\frac{K_1 E}{w}}\cos.^{-1}\frac{k}{K_2 s_1}$$

Both ends flat:

For tension: $$\frac{l}{r} = 4\sqrt{\frac{K_1 E}{w}}\cos.^{-1}\frac{k}{K_2 w}.$$

For compression: Same as both ends fixed.

C. T. PURDY, M. Am. Soc. C. E.—This is an extremely practical subject for engineers who have many columns to dimension or design. In the Astoria Hotel there are about 3 600 separate columns, each of which had to be especially considered. Ten minutes given to each would require 600 hours for all of them. In the speaker's practice every column is analyzed; if there are eccentric loads the exact effects must be determined, and it is therefore important that the methods should be such as will secure the greatest amount of work in the least possible time. Mr. Purdy.

Very few who write regarding column designing realize or appreciate the importance of eccentricity of loading. The speaker does not refer to bridge designing so much as to building work. Most of the columns in buildings are short, and therefore the element of eccentric loading becomes the more important one. The effect of the direct load in such columns is mostly a crushing strain, while the element of bending is of least importance. The effect of eccentricity is rarely lacking, for an even loading is exceptional. From the practical point of view, therefore, the speaker believes that the author's first statement is the most notable one in the paper.

There is great need for a thorough series of experiments on eccentrically loaded columns, but the task is too great to be undertaken by a private engineer. So far as the speaker knows, only two practical tests of eccentrically loaded columns have been made.

WILLIAM CAIN, M. Am. Soc. C. E. (by letter).—This is a valuable paper, not only for a very clear presentation of the theory affecting the practical column, but for the numerous diagrams comparing the theory, supposed to apply to crippling loads approximately, with the results of 1 789 tests of columns by various experimenters. Mr. Cain.

Assuming the neutral line of the column under eccentric loading to be a parabola, the author's equations (5), (7) and (8), follow logically. The writer has called attention* to the propriety of using the parabola approximately, for the neutral line, by comparing results with the exact formula and finding them identical when $\frac{l}{r}$ was

* "Theory of the Ideal Column," *Transactions*, Am. Soc. C. E., Vol. xxxix, p. 120.

Mr. Cain. small. The agreement is still more plainly seen when the author's equation

$$\Delta = \frac{P\, l^2\, e}{8EI - \frac{5}{6} P\, l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

for the deflection at the center of the column pivoted at the ends, from the primitive axis, is compared with the exact result for a prismatic homogeneous column,*

$$\Delta = e \left[\sec. \left(\frac{1}{2} \frac{l}{r} \sqrt{\frac{P}{A\,E}} \right) - 1 \right]$$

Putting $\theta = \frac{1}{2} \frac{l}{r} \sqrt{\frac{P}{A\,E}}$, the development of sec. θ for those values of $\frac{l}{r}$ that give a converging series, is found by the differential calculus to be

$$\text{Sec. } \theta = 1 + \frac{\theta^2}{2} + \frac{5}{24}\theta^4 + \frac{61}{720}\theta^6 + \ldots\ldots$$

On neglecting θ^6 and higher powers of θ, the last formula gives approximately,

$$\Delta = \frac{e}{8} \frac{l^2}{r^2} \frac{P}{A\,E} \left[1 + \frac{5}{48} \frac{l^2}{r^2} \frac{P}{A\,E} \right]$$

which is sensibly correct when θ is a comparatively small fraction, and incorrect otherwise.

But this is exactly equal to Equation (2), if the numerator is divided by the denominator in the right member and the division carried out to two terms.

Hence, the assumption that the neutral line is approximately a parabola for small values of $\frac{l}{r}$ is again proved. The result is a closer approximation, likewise, than that given in "Theory of the Ideal Column" where the writer neglected terms containing θ^4 and higher powers of θ.

Of course, the approximate development for Δ could not be used for most of the values of $\frac{l}{r}$ given on the author's diagrams, and it cannot be utilized, therefore, in deducing the equivalents of (7) and (8); in fact, there would be no gain in doing so, as the author's formulas are briefer than the results corresponding.

The most compact formula is the exact one given by Ernst F. Jonson, Assoc. M. Am. Soc. C. E., and which, adopting the author's notation, is,

$$\frac{l}{r} = \sqrt{\frac{E}{f_a}} \cos.^{-1} \left(\frac{c\,e}{r^2} \frac{f_a}{f_b} \right)$$

---

* See Mr. Marston's Formula (1); *Transactions*, Am. Soc. C. E., Vol. xxxix, p. 113.

Mr. Cain.

where $f_b = (F_c - f_d)$ for the extreme fiber on the concave side and $f_b = (-F_t + f_d)$ for the extreme fiber on the convex side, F_t being $+$ when compressive, $-$ when tensile (page 350).

Recurring now to "the application of the formulas in practice" (page 414), there is room for difference of opinion as to basing the factor of safety on the lower-limit curves of the diagrams, especially where the lower-limit curves pass below all the plotted positions as given by experiment.

If very small factors of safety are used, then, undoubtedly, the lower-limit curves should give the basis for the formulas to use in practice, or disaster might follow on increasing the loads, as the column in question might have a crippling strength near that given by the lower-limit curve. Thus, if the factor of safety was 2 and the loads were eventually doubled (as has happened on a number of railways in this country in the past), the column would certainly be regarded as dangerous if the factor 2 was based on the average curve passing somewhat centrally through the points on the diagram given by experiment.

But if the factor is 4 or 5, it seems safe to take the average curve, even if the loads are to be doubled in twenty or thirty years. Some of the columns will thus have factors of safety above that given and others below; so it would seem that the greatest economy should be secured by basing the safe curve upon the average curve.

If there is any merit in this contention, then the objection is more marked when for flat-ended columns "the curve indicating the load at which tensile stress commences" falls far below the results for experiments, as in some of the diagrams. In fact, failure does not begin for such a load for any given column. The case is similar in one respect to the voussoir arch without mortar joints, when the line of the centers of pressure falls outside the middle third. No tension is exerted on a joint where this occurs, but the compressive stresses are distributed according to a uniformly increasing law, and the material may be strong enough to withstand the maximum stress induced. Similarly, in the column, where the resultant force is not too near the edges of the end sections, the maximum stresses in the extreme fibers may not be too great for strength or stability. Some allowance, however, should be made here for such cases, and the results of experiments should be the main guide.

Mr. Mayer.

Joseph Mayer, M. Am. Soc. C. E. (by letter).—This paper contains a valuable collection of tests of columns, and is an interesting effort to derive, by means of mathematical reasoning, formulas giving the fiber stresses in actual columns.

The author appreciates the necessity of introducing empirical coefficients in his formulas, for the purpose of allowing for eccentricity

Mr. Mayer. of loading, initial curvature, etc., to make them agree with the tests. One of the purposes of the paper is to find formulas or curves for the ultimate strengths of columns. The curves obtained deviate very far from the ultimate strengths of short columns shown by the tests. The engineer cares most for the strength of columns of a length of less than 150 radii of gyration, and the curves of ultimate strength ought to be approximately correct for these columns. For all kinds of steel, wrought-iron and wooden columns it is possible to find straight lines which agree better with the actual ultimate strength shown by the tests of columns of less than 150 radii of gyration than the author's complicated curves. These straight lines are more convenient for use and more easily remembered.

The engineer desires to use dead loads per square inch equal to half of those producing stresses equal to the elastic limit. For very short steel or wrought-iron columns the elastic limit is about 0.6 of the ultimate strength. The proper dead load is, therefore, 0.3 of the ultimate strength, and the ultimate strength is about equal for tension and compression. For longer columns, the elastic limit and ultimate strength are nearly identical. A curve which starts with 0.3 of the ultimate strength for columns of no length and approaches half the ultimate strength of long columns answers the purposes of the engineer. A straight line, of the equation $u = a - b \frac{l}{r}$, where $a = 0.3$ of the ultimate strength of the steel or wrought iron of which the column is made, $b = \frac{a}{250}$, will give the proper unit strain u for the dead load of columns of less than 150 radii of gyration, with as much accuracy as the uncertainty of the case allows. More complicated formulas would only be justified if they agreed more closely with the tests than do these simple formulas, which is not the case for columns of moderate length, which are alone used in practice.

The formulas become, therefore, for iron $14\,000 - 56 \frac{l}{r}$; for medium steel $18\,000 - 72 \frac{l}{r}$.

The usual straight-line formulas for timber are as accurate as the author's formulas. While the writer, therefore, cannot agree with the author in his conclusions, he highly appreciates the services rendered by the latter in collecting the tests and presenting his views in such a lucid and simple manner.

Mr. Frye. ALBERT I. FRYE, M. Am. Soc. C. E. (by letter).—The author's assumption that a deflected column will assume a parabolic curve when subjected to an eccentric loading is fairly correct and allowable. It will be, as he states, somewhere between a curve of sines and a circle, depending on the amount of eccentricity.

Mr. Frye.

The writer found, several years ago, that for a beam loaded with a parabolic load W, in Fig. 53, the bending moments and the deflections will be nearly proportional to the sines of the respective angles; the ends of the beam corresponding to 0°, and the center of the beam to 90°. From this he obtained practically Euler's formula,

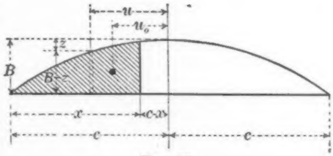


FIG. 53.

$$P = \frac{\pi^2 E I}{l^2}.$$

This demonstration is as follows:

Let W = total weight on the beam;
W_o = weight to left of section considered;
B = maximum intensity of load;
c = half span $= \frac{l}{2}$;
x = distance from end to section considered;
u_o = distance to center of gravity of load to left of section considered.

then, $$u_o = \frac{\int_{c-x}^{c} u\,(B - z)\,d\,u}{\int_{c-x}^{c} (B - z)\,d\,u} = \frac{12\,c^2 - 12\,c\,x + 3\,x^2}{12\,c - 4\,x}, \quad\ldots\ldots\ldots \text{(1)}$$

and when $$x = c,\ u_o = \frac{3}{8}c \ldots\ldots\ldots\ldots \text{(1}a\text{)}$$

$$W_o = \int_{c-x}^{c} (B - z)\,d\,u = \frac{B\,(3\,c\,x^2 - x^3)}{3\,c^2} \ldots\ldots\ldots\ldots \text{(2)}$$

and when $$x = c,\ W_o = \frac{2}{3}\,B\,c$$

or $$W = \frac{4}{3}\,B\,c \ldots\ldots\ldots\ldots \text{(2}a\text{)}$$

The moment at any section distant x from the end of the beam equals:

$$M_x = \frac{2}{3}B\,c\,x - \frac{B\,(3\,c\,x^2 - x^3)}{3\,c^2}\,(u_o + x - c).$$

Substituting the value of u_o in (1) and reducing,

$$M_x = \frac{B\,(8\,c^3\,x - 4\,c\,x^3 + x^4)}{12\,c^2} \ldots\ldots\ldots\ldots \text{(3)}$$

and when $$x = c,\ M = \frac{5}{12}\,B\,c^2 \ldots\ldots\ldots\ldots \text{(3}a\text{)}$$

Mr. Frye. The angle of slope of the deflected beam at any point distant x from the end equals:

$$i = \frac{1}{E\,I}\int M\,d\,x$$

$$= \frac{B}{12\,c^2\,E\,I}\left(4\,c^3\,x^2 - c\,x^4 + \frac{x^5}{5}\right) + k.$$

To find the value of the constant k,

when $x = c, \quad i = o,$

therefore, $$k = -\frac{B}{12\,c^2\,E\,I}\left(\frac{16\,c^5}{5}\right)$$

and $$i = \frac{B}{12\,c^2\,E\,I}\left(4\,c^3\,x^2 - c\,x^4 + \frac{x^5}{5} - \frac{16\,c^5}{5}\right) \qquad (4)$$

when $$x = 0, \quad i = \frac{4}{15}\,\frac{B\,c^3}{E\,I} \qquad (4a)$$

The deflection of the beam at any point distant x from the end equals:

$$v = \int i\,d\,x = \frac{B}{12\,c^2\,E\,I}\int\left(4\,c^3\,x^2 - c\,x^4 + \frac{x^5}{5} - \frac{16\,c^5}{5}\right)d\,x$$

$$= \frac{B}{12\,c^2\,E\,I}\left(\frac{4\,c^3\,x^3}{3} - \frac{c\,x^5}{5} + \frac{x^6}{30} - \frac{16\,c^5\,x}{5}\right) + k_1$$

when $x = 0, \quad v = 0$; therefore $k_1 = 0$,

and $$v = \frac{B}{12\,c^2\,E\,I}\left(\frac{4\,c^3\,x^3}{3} - \frac{c\,x^5}{5} + \frac{x^6}{30} - \frac{16\,c^5\,x}{5}\right) \qquad (5)$$

when $$x = c, \quad v = -\frac{61}{360}\,\frac{B\,c^4}{E\,I} \qquad (5a)$$

Dividing the half span c corresponding to 90° into six parts of 15° each, and calculating from (3) the bending moments at each section; calling the bending moment at the center of the beam unity, and likewise, using (5) for deflections; calling the deflection at the center of the beam unity, and comparing these results with the curve of sines, and the parabolic curve, gives the results shown in Table No. 6.

TABLE No. 6.

	0°	15°	30°	45°	60°	75°	90°
Bending moments	0	0.263	0.506	0.712	0.869	0.966	1
Deflections	0	0.262	0.501	0.707	0.866	0.966	1
Curve of sines	0	0.259	0.500	0.707	0.866	0.966	1
Parabolic curve	0	0.306	0.556	0.750	0.889	0.972	1

It will be seen on examination of Table No. 6 that the bending moments and the deflections correspond very nearly with the curve of sines of the theoretical deduction of Euler. As stated by the author, the actual curve will depart from the above ideal upon the application of an eccentric load on the column, and tend to conform more closely to the circle. By calculation, the writer has found that, for a deflection of unity at the center, in a column 480 units in length, the circle and the parabola are practically coincident. This would tend to show that, so far as the curve of the column is concerned, the assumption is on the basis of a preponderance of eccentric loading, since the latter, taken abstractly, would curve the column to the arc of a circle, or practically a parabola. The assumption, therefore, is on the side of safety. Mr. Frye.

The bending moment at the center section of the column—Equations (3*a*) and (2*a*)—equals:

$$M = \frac{5}{12} B c^2 = \frac{5}{8} W_o c$$

$$= \frac{5}{32} W l \quad \ldots\ldots\ldots\ldots (6)$$

If $$M = \frac{f I}{y}, \text{ then } f = \frac{5 W l y}{32 I} \quad \ldots\ldots\ldots\ldots (6a)$$

The deflection at the center section of the column—Equations (5*a*) and (2*a*)—equals:

$$v = -\frac{61}{360} \frac{B c^4}{E I} = -\frac{61}{240} \frac{W_o c^3}{E I}$$

$$= -\frac{61}{3840} \frac{W l^3}{E I} \quad \ldots\ldots\ldots\ldots (7)$$

From (6*a*), $W = \dfrac{32 f I}{5 l y}$, therefore $v = -\dfrac{61}{600} \dfrac{f l^2}{E y}$ .... (7*a*)

Substituting this value of v in the general equation $M = P v$, and, omitting the minus sign, there is obtained:

$$P = \frac{M}{v} = \frac{\frac{f I}{y}}{\frac{61}{600} \frac{f l^2}{E y}} = \frac{600}{61} \frac{E I}{l^2} \quad \ldots\ldots\ldots\ldots (8)$$

$$\frac{600}{61} = 9.836, \text{ whereas } \pi^2 = 9.87.$$

This proves that a lateral parabolic load W will produce practically the same curvature in a column as a direct concentric load P.

The paper is very valuable in presenting such a collection of tests, and so closely interpreting the causes of failure.

Mr. Moncrieff. J. M. MONCRIEFF, M. Am. Soc. C. E. (by letter).—The precise character of the curve assumed by a bent column is stated in the paper to be a matter of small importance, and the writer will go further than this, it is of no practical importance. This is shown clearly in the paper in the comparison between the formula (Euler's) for an ideal column and the writer's formula based on the assumption that the curve is parabolic.

As a matter of fact, the irregularities in the condition of the material, and in the conditions usually met with in practice, will cause the curve to be neither a curve of sines nor a parabola.

Mr. Ernst F. Jonson's modification or transformation of Mr. Marston's formula does not remove it from the category of those formulas which are awkward and inconvenient to use in practice.

A comparison, in actual working out, between Mr. Jonson's formula and that of the writer, will certainly convince anyone that the simplicity of the latter, in practical work, is very much greater, and also that it is, in every practical sense, quite as correct. There is nothing to be gained by introducing unnecessary mathematical refinements which have little or no influence on the results.

Mr. Jonson states that:

"Apart from its mathematical form, the foregoing formula (Mr. Jonson's) differs from that of the author in that it makes the maximum load a function, not of the ultimate strength of the material, but of the proportional limit."

There is no difference between the formulas in this respect, as each is based on the primary assumption that the material dealt with is perfectly elastic.

Mr. Jonson probably means that he, personally, does not approve of taking the maximum load to be a function of the ultimate strength of the material.

In the paper, the values given to F_c, in making the comparison between the results of calculation and the records of tests carried to ultimate column failure, are not values of the ultimate compressive strength of the material, but are only moduli of column rupture (see lines 18 to 28 of page 369).

The writer is very glad to have Professor Cain's appreciation of the theory of the paper and of the formulas deduced therefrom. His suggestion, that the greatest economy should be secured by basing the safe load upon the average tests, however, does not appear to the writer to be a rational one.

Reference to the diagram (Fig. 34) showing the tests of wrought-iron flat-ended columns, will show that, between the proportions of $\frac{l}{r} = 40$ to 490, a considerable number of results falls close to the writer's lower-limit curves, some being below the curves; and, unless

doubts can be cast upon the care and accuracy of the experimenters who carried out these tests, it must be recognized that the lesson which these results teach is, that, in columns of these proportions, greater ultimate strength cannot be safely counted upon. Mr. Moncrieff.

Why, then, should engineers soothe their minds with the hope that the strength may happen to be considerably higher? No economy can result from such a procedure.

The writer is much obliged to Mr. O'Hanly for the trouble he has taken in checking over some of the computed deflections. The results caused the writer to have an entirely independent check made of the working out of the calculated deflections, and, while it was found that Mr. O'Hanly's figures could not be entirely accepted, his criticism has resulted in several errors being discovered and corrected.*

In reply to Mr. Joseph Mayer's remarks, the writer desires to disclaim that "one of the purposes of the paper is to find formulas or curves for the ultimate strengths of columns." The writer's purpose was to develop a practical and simple theory of column strength under central or eccentric loads, on the lines stated in the beginning of the paper, and, in order to justify the theory and its resulting formulas, they were compared with the results of practical experiment. The results of the comparison scarcely warrant Mr. Mayer's criticism that:

"The curves obtained deviate very far from the ultimate strengths of short columns shown by the tests. The engineer cares most for the strength of columns of a length of less than 150 radii of gyration, and the curves of ultimate strength ought to be approximately correct for these columns."

Attention is called to the diagrams of tests of the materials which are in most common use to-day, *i. e.*, wrought iron and mild steel.

Fig. 22; Containing all the Chief Tests of Wrought-Iron Columns with Round and Pivoted Ends.

At $\frac{l}{r}$ = about 46, lower-limit curve is 1 600 lbs. per square inch below lowest tests.

" 76,	"	"	1 100 "	per square inch below lowest tests.
" 112,	"	"	900 "	per square inch above lowest tests.
" 131,	"	"	1 000 "	per square inch below lowest tests.
" 145,	"	"	200 "	per square inch above lowest tests.
" 170,	"	"	2 000 "	per square inch above lowest tests.

* This refers to the discussion by J. L. Power O'Hanly, M. Am. Soc. C. E., which was published in *Proceedings* for August, 1900, but is not reproduced here, the revision of the paper making it unnecessary.

Mr. Moncrieff. *Fig. 24; Eccentric Loading of Wrought-Iron Bars.*

From ratio $\frac{l}{r} = 44$ to ratio $\frac{l}{r} = 342$ } The tests are included between two curves which, at their point of maximum deviation, are 3 700 lbs. per square inch apart.

Fig. 25; Eccentric Loading of Wrought-Iron Bars.

From ratio $\frac{l}{r} = 46$ to ratio $\frac{l}{r} = 183$ } The tests lie between pairs of curves which are only 600 lbs. per square inch apart at the maximum point.

Fig. 26; Tests of Fixed-Ended Wrought-Iron Columns.

For $\frac{l}{r}$ = 43,	lower-limit	curve is	400	lbs.	per square inch below lowest test.
70,	"	"	200	"	per square inch below lowest test.
105,	"	"	1 600	"	per square inch below lowest test.
131,	"	"	400	"	per square inch below lowest test.
141,	"	"	1 200	"	per square inch above lowest test.
147,	"	"	2 500	"	per square inch above lowest test.
175,	"	"	400	"	per square inch above lowest test.
217,	"	"	1 300	"	per square inch above lowest test.
221,	"	"	1 400	"	per square inch above lowest test.

Fig. 34; Containing all the Tests of Flat-Ended Wrought-Iron Columns. In this diagram the lower-limit curve from $\frac{l}{r} = 40$ to $\frac{l}{r} = 490$ gives a very fair line for the lowest tests for the great range just mentioned.

Between $\frac{l}{r} = 40$ and $\frac{l}{r} = 150$ there is one test, *i. e.*, at 125 about, which is 3 600 lbs. per square inch below the writer's curve, and there are several others below the curve between 40 and 150, but the maximum of these is 2 500 lbs. per square inch below the curve.

Does Mr. Mayer mean that the writer has overestimated the ultimate strength of these columns up to 150 radii in length? If so, how would he regard the estimate of strength recommended by Professor Cain, *i. e.*, the average of all tests, which would be very much higher than the writer's curve?

Fig. 38; Containing all the Tests of Hinged-Ended Wrought-Iron Columns.—It is unnecessary to take figures for this diagram. The lower-

limit curve drawn by the writer runs almost through the whole of the lowest tests, with one exception, *i. e.*, that at $\frac{l}{r} = 107$, and the agreement is not only for tests up to 150 radii in length, but extends from 35 up to 500. Mr. Moncrieff.

Fig. 39; Containing all Tetmajer's Tests of Mild-Steel Columns with Pivoted Ends.—From $\frac{l}{r} = 30$ to $\frac{l}{r_1} = 355$, the writer fails to find the great divergence of the lower-limit curve from the tests, referred to by Mr. Mayer.

Fig. 40; Flat-Ended Mild-Steel Columns.—From $\frac{l}{r} = 25$ to $\frac{l}{r} = 325$, the lower-limit curve drawn cannot truly be said to deviate, to any great extent, from the tests.

Fig. 41; Mild-steel; Hinged Ends.—Can Mr. Mayer, or any other engineer, draw through these tests a straight line which will be a fair guide in practice? If a column at $\frac{l}{r} = 118$ has a strength of 45 000 lbs. per square inch, why should another column at $\frac{l}{r} = 125$ have a strength of only about 32 000 lbs. per square inch?

Why should columns of from 55 to 110 radii long be credited with the strengths shown by the tests on this diagram, when columns very little longer, of the same material and by the same experimenter, exhibit such a serious and sudden drop in the test results?

What straight and single-line formula can give a rational idea of such vagaries as these?

Are Mr. Mayer's straight-line formulas to be applied to columns with pivoted ends, hinged ends, flat ends, or fixed ends, indiscriminately?

The idea of the paper was to give a general theory, with simple resulting formulas, which "should have a wide range of application to cover the conditions met in engineering practice."

No straight-line formula can fulfil this condition, or be applied to the varying cases of eccentric loading, or to cases where there are side loads or exceptional forms of cross-section.

In conclusion, the writer heartily thanks those who have discussed the paper for their very kindly expressions of approval.